A study of Besov-Lipschitz and Triebel-Lizorkin spaces using non-smooth kernels

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Abstract. We consider the problem of characterising Besov-Lipshitz and Triebel-Lizorkin spaces using kernels with limited smoothness and decay. This extends the work of H.-Q. Bui et al in [**4**] and [**5**] from kernels in $\mathcal{S}$ to more general kernels, including the Poisson kernel. We overcome the difficulty of defining the convolution of a general kernel with a distribution by using the concept of a bounded distribution introduced by E. Stein [**12**]. The characterisations we obtain are valid for the full range of indices $0 < p, q \leqslant \infty$ and $\alpha \in \mathbb{R}$.

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Contents

CHAPTER 1

Introduction

The Besov-Lipschitz and Triebel-Lizorkin spaces have a long history and their properties have been well studied. As with many function spaces, the Besov-Lipschitz and Triebel-Lizorkin spaces come in two varieties, the homogeneous versions, denoted by $\dot{B}^{\alpha}_{p,q}$ and $\dot{F}^{\alpha}_{p,q}$ respectively, and the inhomogeneous versions, denoted by $B^{\alpha}_{p,q}$ and $F^{\alpha}_{p,q}$. These types of spaces arise naturally when one considers the problem of measuring the smoothness of a distribution. There are various ways of measuring the smoothness of a distribution, one way is to ask for a generalisation of the classical homogeneous Sobolev spaces $\dot{W}^{p,k}$ where, for $k \in \mathbb{N}$ with $k > 0$, we define

$$\dot{W}^{p,k} = \{f \in \mathcal{S}' \mid D^{\kappa} f \in L^p \quad \forall\ |\kappa| = k\}.$$

The homogeneous Triebel-Lizorkin space, $\dot{F}^{\alpha}_{p,q}$, satisfies this requirement in the sense that $\dot{F}^{k}_{p,2} = \dot{W}^{p,k}$ whenever $k \in \mathbb{N}$ with $k > 0$, and $1 < p < \infty$. We also remark that the restriction of $f \in \dot{W}^{p,k}$ on $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$ belongs to a certain Besov-Lipschitz space; see [**1**]. These results hint at the important applications of Triebel-Lizorkin and Besov-Lischitz spaces to the study of partial differential equations (PDEs), as the Sobolev spaces are widely used in the theory of PDEs.

The scale of spaces, $\dot{F}^{\alpha}_{p,q}$, also includes other important function spaces such as the Hardy space, H^p, where we define, using the characterisation of Fefferman and Stein (see [**12**, pg 91]),

$$H^p = \left\{ f \in \mathcal{S}' \;\middle|\; \|f\|_{H^p} = \Big\| \sup_{t>0} |\phi_t * f| \Big\|_p < \infty \right\}$$

with $\phi \in \mathcal{S}$ such that $\widehat{\phi}(0) > 0$. In this case it can be proven that we have the equivalence $\dot{F}^{0}_{p,2} = H^p$; see [**2**]. Throughout this thesis we restrict our attention to the homogeneous Besov-Lipchitz and Triebel-Lizorkin spaces. We refer the interested reader to [**15**] and [**11**] for more on the history and origins of the Besov-Lipschitz and Triebel-Lizorkin spaces.

The definition of the spaces $\dot{B}^{\alpha}_{p,q}$ and $\dot{F}^{\alpha}_{p,q}$ will require the use of a kernel $\varphi \in \mathcal{S}$ satisfying certain conditions. We defer the precise definition until Section 2 but we note here that φ is required to be band limited, that is, $\widehat{\varphi}$ has compact support. This restriction immediately removes the possibility of directly using classical kernels such as the Gaussian and Poisson kernels in the definitions of $\dot{B}^{\alpha}_{p,q}$ and $\dot{F}^{\alpha}_{p,q}$, as neither of these kernels are band limited. We remark however that with a little work it is possible to characterise the spaces $\dot{B}^{\alpha}_{p,q}$ and $\dot{F}^{\alpha}_{p,q}$ using the Gaussian and Poisson kernels. The Gaussian kernel case is contained in Theorem 1.3, while the Poisson kernel case, for $p \geqslant 1$, is proven in Theorem 4.5.

More generally, in [**4**] and [**5**] the authors show that to characterise the Besov-Lipschitz and Triebel-Lizorkin spaces we only need the kernel $\varphi \in \mathcal{S}$ to satisfy a Tauberian condition and a moment condition; see Theorem 1.3. This thesis is concerned with the problem of removing the restriction $\varphi \in \mathcal{S}$. We show that even without this assumption, it is still possible obtain results similar to those presented in [**4**] and [**5**].

We now present a brief outline of this thesis. This first introductory chapter will set the notation, introduce the Besov-Lipschitz and Triebel-Lizorkin spaces, and include some useful properties of these spaces. It will also include a section on bounded distributions, a class of (tempered) distributions for which it is possible to define the convolution with a general kernel, something not possible for an arbitrary $f \in \mathcal{S}'$.

The remainder of this thesis deals with the problem of characterising the spaces $\dot{B}^{\alpha}_{p,q}$ and $\dot{F}^{\alpha}_{p,q}$ using a general kernel ψ. In Chapter 2 we prove the necessary direction, that is, we prove inequalities of the form

$$\left(\sum_{j\in\mathbb{Z}} \left(2^{j\alpha}\|\psi_j * f\|_{H^p}\right)^q \right)^{1/q} \leqslant C\|f\|_{\dot{B}^{\alpha}_{p,q}};$$

see Theorem 2.1 and Theorem 2.7.

Chapter 3 concerns the more difficult problem of finding sufficient conditions on $f \in \mathcal{S}'$ to belong to $\dot{B}^{\alpha}_{p,q}$ and $\dot{F}^{\alpha}_{p,q}$, that is, we consider inequalities of the form

$$\|f\|_{\dot{B}^{\alpha}_{p,q}} \leqslant C_2\left(\sum_{j\in\mathbb{Z}} \left(2^{j\alpha}\|\psi_j * f\|_p\right)^q \right)^{1/q}.$$

The problem of finding a pointwise definition for the convolution $\psi * f$, for $f \in \mathcal{S}'$ and ψ not in $\mathcal{S}$, will prove difficult to overcome and we will have to resort to proving maximal function characterisations; see Theorem 3.5. However, in a few special cases, we can provide sufficient conditions without the use of maximal functions, though we have been thus far unable to complete the proof in the case $0 < p < 1$; see Theorem 3.7 and Theorem 3.10.

In the final chapter we bring together Chapters 2 and 3 and present characterisations of $\dot{B}^{\alpha}_{p,q}$ and $\dot{F}^{\alpha}_{p,q}$ with very general conditions on our kernel; see Theorem 4.1 for the Besov-Lipschitz case and Theorem 4.3 for the Triebel-Lizorkin case.

1. Definitions and Notation

All functions appearing in this thesis shall be measurable, have domain $\mathbb{R}^n$, and be complex-valued unless stated otherwise. We will reserve the letter n to denote the dimension of the Euclidean space $\mathbb{R}^n$. More general domains will not be considered here. The letter C will denote a constant independent of the variable quantity (usually a distribution f) that will change from line to line. We use the usual definitions $\mathbb{N} = \{0, 1, 2, ...\}$ for the natural numbers and $\mathbb{Z}$ for the integers.

For a multi index $\kappa = (\kappa_1, \kappa_2, ..., \kappa_n)$, $\kappa_j \in \mathbb{N}$, we define $|\kappa| = \kappa_1 + \kappa_2 + ... + \kappa_n$ and

$$D^\kappa = \partial_{x_1}^{\kappa_1} \partial_{x_2}^{\kappa_2} ... \partial_{x_n}^{\kappa_n}$$

where ∂_{x_j} is the partial derivative with respect to x_j. We also use the obvious notation $x^\kappa = {x_1}^{\kappa_1} {x_2}^{\kappa_2} ... {x_n}^{\kappa_n}$ for $x = (x_1, ..., x_n) \in \mathbb{R}^n$. The open ball in $\mathbb{R}^n$ centered at x of radius r shall be denoted by $B(x, r)$. For $\alpha \in \mathbb{R}$ we use the notation $[\alpha]$ to denote the integer part of α, i.e.

$$[\alpha] = \max\{k \in \mathbb{Z} \mid k \leqslant \alpha \ \}$$

We will also make use of the usual definitions,

$$\widetilde{\eta}(y) = \eta(-y),$$

and

$$\tau_y \eta(x) = \eta(x - y).$$

For $0 < p \leqslant \infty$ we define $L^p(\mathbb{R}^n)$ to be the collection of all measurable, complex-valued functions f on $\mathbb{R}^n$ such that

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p},$$

where the integration will be with respect to the Lebesgue measure. We assume the reader is familiar with the usual properties of the Lebesgue integral and L^p spaces such as The Dominated Convergence Theorem, Monotone Convergence Theorem, Fubbini's Theorem, duality, etc. Since all functions will be defined on $\mathbb{R}^n$, we will omit writing $\mathbb{R}^n$ and just use the abbreviation $L^p = L^p(\mathbb{R}^n)$ when we can do so without causing confusion. This comment will also apply to all other function spaces which will appear throughout this thesis.

We will also make use of the closely related l^p spaces. We say the sequence $(a_k)_{k\in\mathbb{Z}}$ of complex numbers belongs to l^p if

$$\left\|(a_k)_{k\in\mathbb{Z}}\right\|_{l^p} = \Big(\sum_{k\in\mathbb{Z}} |a_k|^p \Big)^{1/p} < \infty.$$

We note that the l^p spaces are monotone in the sense that $l^p \subseteq l^q$ whenever $p \leqslant q$. This is a consequence of the easily proven fact that $\|(a_k)_{k\in\mathbb{Z}}\|_{l^q} \leqslant \|(a_k)_{k\in\mathbb{Z}}\|_{l^p}$ for any $p \leqslant q$.

For an open subset X of $\mathbb{R}^n$ and $k \in \mathbb{N}$ we denote by $C^k(X)$ the space of all continuous functions on X which have continuous partial derivatives up to order k. As remarked above when $X = \mathbb{R}^n$ we will use the short hand $C^k = C^k(\mathbb{R}^n)$.

We denote by $\mathcal{S}$ the Schwartz class of infinitely differentiable, rapidly decreasing functions on $\mathbb{R}^n$. The dual of $\mathcal{S}$ will be denoted by $\mathcal{S}'$ and we will refer to elements of $\mathcal{S}'$ as distributions (although they should more correctly be called *tempered* distributions).

The Fourier transform will initially be defined for any $\psi \in L^1$ by

$$\widehat{\psi}(\xi) = \int_{\mathbb{R}^n} \psi(x)\, e^{-2\pi i x\cdot\xi}\, dx.$$

We then extend the Fourier Transform to L^2 and $\mathcal{S}'$ in the usual manner. Throughout this thesis the notation $\widehat{\psi}$ will always denote the Fourier transform of ψ. We assume the reader is familiar with the basic theorems of Fourier analysis on $\mathbb{R}^n$.

For measurable functions ψ and η, we define the convolution $\psi * \eta$ by

$$\psi * \eta(x) = \int_{\mathbb{R}^n} \psi(x-y)\, \eta(y)\, dy,$$

whenever the last integral is defined. We note that for $p \geqslant 1$ we have Young's Inequality

$$\|\psi * \eta\|_p \leqslant \|\psi\|_1 \|\eta\|_p.$$

The convolution can also be extended extended to $\mathcal{S}'$. For $f \in \mathcal{S}'$ and $\phi \in \mathcal{S}$ we define

$$\phi * f(x) = f(\tau_x \widetilde{\phi}).$$

It is well known that the function $\phi * f \in C^\infty$, and moreover, that $\phi * f$ is *slowly increasing* in the sense that there exists $k \in \mathbb{N}$ such that, for every $x \in \mathbb{R}^n$,

$$|\phi * f(x)| \leqslant C(1 + |x|)^k.$$

We note that in general it is not possible to define the convolution of $f \in \mathcal{S}'$ with a general kernel ψ.

Let $0 < t < \infty$. We define the dilation ψ_t by

$$\psi_t(x) = \frac{1}{t^n}\psi(x/t).$$

For a discrete parameter $j \in \mathbb{Z}$ we make the slight abuse of notation and write

$$\psi_j(x) = \psi_{2^{-j}}(x) = 2^{jn}\psi(2^j x).$$

We will use the following result. Let $\psi \in L^1$ with $\widehat{\psi}(0) = 1$. If $g \in L^p \cap C^0$ for $1 \leqslant p \leqslant \infty$ we have, for every $x \in \mathbb{R}^n$,

$$\lim_{t \to 0} \psi_t * g(x) = g(x). \tag{1}$$

A similar result holds when $(1 + |\cdot|)^\lambda \psi(\cdot) \in L^1$ and $(1 + |\cdot|)^{-\lambda} g(\cdot) \in L^\infty \cap C^0$, since then the convolution $\psi * g$ is well-defined. We remark that (1) also applies when $\psi \in \mathcal{S}$ and $g \in \mathcal{S}'$, since then again the convolution $\psi * g$ is well-defined. However the convergence is now in $\mathcal{S}'$.

1.1. Vanishing Moments. Let $k \in \mathbb{Z}$. We say $\psi \in L^1$ has k *vanishing moments* if for every $|\kappa| \leqslant k$ we have

$$\int_{\mathbb{R}^n} x^\kappa \, \psi(x) \, dx = 0. \tag{2}$$

If $k < 0$ we make the convention that no moment condition is required. It is easy to see that if $\psi \in \mathcal{S}$ then $D^\kappa \psi$ has $|\kappa| - 1$ vanishing moments, thus taking derivatives gives vanishing moments. The following theorem shows that in fact the converse is also true.

THEOREM 1.1. *Suppose $\mu \in \mathcal{S}$ has k vanishing moments. Then for every $|\kappa| = k + 1$ there exists $\mu^\kappa \in \mathcal{S}$ such that*

$$\mu = \sum_{|\kappa| = k+1} D^\kappa \mu^\kappa.$$

PROOF. See Theorem B.1. $\square$

We remark that if $\widehat{\psi}$ vanishes in a neighbourhood of the origin then ψ has infinite vanishing moments. This in turn implies

$$\int_{\mathbb{R}^n} \rho(x) \, \psi(x) \, dx = 0$$

for any polynomial ρ.

1.2. Tauberian Condition. Let $\psi \in L^1$. We say ψ satisfies the *Tauberian condition* if for every $|\xi| = 1$ there exists a $c > 0$ and $0 < 2\sigma \leqslant \varrho < \infty$ such that for each $\sigma < t < \varrho$

$$|\widehat{\psi}(t\xi)|^2 \geqslant c > 0. \tag{3}$$

The Tauberian condition implies that the family of functions $\big(\widehat{\psi}(2^{-j}\xi)\big)_{j \in \mathbb{Z}}$ does not simultaneously vanish for every $|\xi| > 0$. This observation will allow us to construct a function η such that for every $|\xi| > 0$ we have the equality

$$\sum_{j \in \mathbb{Z}} \widehat{\psi}(2^{-j}\xi) \widehat{\eta}(2^{-j}\xi) = 1.$$

Formally, by multiplying both sides by $\widehat{\mu}$ and taking the inverse transform, this equality would then give the representation

$$\mu(x) = \sum_{j\in\mathbb{Z}} \psi_j * \eta_j * \mu(x); \tag{4}$$

see Chapter 3, Section 1 for a more detailed exposition. Equation (4) is a version of the Calderón reproducing formula which will prove crucial in our study of the Besov-Lipschitz and Triebel-Lizorkin spaces. The Calderón type formulas we will need are not the most general that can be obtained, we refer the reader to [**9**] for certain generalisations. The versions which we require will be proven in Chapter 2, Section 1 and Chapter 3, Section 1.

Finally, we urge the reader to review the inequalities given in Proposition B.2 and Proposition B.3, as these inequalities will simplify the results presented in later chapters.

2. Besov-Lipschitz and Triebel-Lizorkin Spaces

As mentioned in the introduction, we only consider the homogeneous Besov-Lipschitz and Triebel-Lizorkin spaces. Thus we will usually omit the homogeneous and just refer to the spaces $\dot{B}^{\alpha}_{p,q}$ and $\dot{F}^{\alpha}_{p,q}$ as the Besov-Lipschitz and Triebel-Lizorkin spaces. The definition of the spaces $\dot{B}^{\alpha}_{p,q}$ and $\dot{F}^{\alpha}_{p,q}$ will require a kernel $\varphi \in \mathcal{S}$ satisfying, for any $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^{-j}\xi)\widehat{\varphi}(2^{j}\xi) = 1 \tag{5}$$

and moreover

$$\operatorname{supp}\ \widehat{\varphi} \subseteq \{1/2 \leqslant |\xi| \leqslant 2\}. \tag{6}$$

See Appendix A for an example of such a φ. For the remainder of this thesis we fix a $\varphi \in \mathcal{S}$ satisfying (5) and (6).

We can now define the homogeneous Besov-Lipschitz space, $\dot{B}^{\alpha}_{p,q}$, as follows. For $0 < p, q \leqslant \infty$, $\alpha \in \mathbb{R}$, and any distribution $f \in \mathcal{S}'$, we define

$$\|f\|_{\dot{B}^{\alpha}_{p,q}} = \Big(\sum_{j\in\mathbb{Z}} (2^{j\alpha}\|\varphi_j * f\|_p)^q\Big)^{1/q}$$

and

$$\dot{B}^{\alpha}_{p,q} = \{f \in \mathcal{S}' \mid \|f\|_{\dot{B}^{\alpha}_{p,q}} < \infty\ \}.$$

Similarly, for $\alpha \in \mathbb{R}$, $0 < q \leqslant \infty$, and $0 < p < \infty$, we let

$$\|f\|_{\dot{F}^{\alpha}_{p,q}} = \Big\|\Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}|\varphi_j * f|\big)^q\Big)^{1/q}\Big\|_p$$

and define the homogeneous Triebel-Lizorkin space, $\dot{F}^{\alpha}_{p,q}$, by

$$\dot{F}^{\alpha}_{p,q} = \{ f \in \mathcal{S}' \mid \|f\|_{\dot{F}^{\alpha}_{p,q}} < \infty \}.$$

Note that the assumption (6) on φ implies that φ has infinite vanishing moments. Thus, for any polynomial ρ, we have $\varphi * \rho = 0$ and as a result

$$\|\rho\|_{\dot{B}^{\alpha}_{p,q}} = 0,$$

hence $\|\cdot\|_{\dot{B}^{\alpha}_{p,q}}$ is not a norm. However, if we consider elements of $\dot{B}^{\alpha}_{p,q}$ modulo polynomials, then $\|\cdot\|_{\dot{B}^{\alpha}_{p,q}}$ does form a norm (quasi-norm if $0 < \min\{p,q\} < 1$). Moreover, if we regard $\dot{B}^{\alpha}_{p,q}$ as a subset of $\mathcal{S}'/P$ (where P is the set of all polynomials), then $\dot{B}^{\alpha}_{p,q}$ is a Banach space (quasi-Banach space if $0 < \min\{p,q\} < 1$); see [**2**]. A similar comment applies for the Triebel-Lizorkin space $\dot{F}^{\alpha}_{p,q}$. For the remainder of this thesis we will slightly abuse notation by refering to the quasi-norms, $\|\cdot\|_{\dot{B}^{\alpha}_{p,q}}$ and $\|\cdot\|_{\dot{F}^{\alpha}_{p,q}}$, as norms.

We also remark that different choices of φ satisfying (5) and (6) lead to the same spaces $\dot{B}^{\alpha}_{p,q}$ and $\dot{F}^{\alpha}_{p,q}$ with equivalent norms. We refer the reader to the work of J. Peetre, [**10**], for a proof of this fact.

We note the following, mostly elementary, properties of the (homogeneous) Besov-Lipschitz and Triebel-Lizorkin spaces.

THEOREM 1.2. *Let $0 < p < \infty$, $0 < q \leqslant \infty$ and $\alpha \in \mathbb{R}$.*

(i) For any $f \in \mathcal{S}'$ we have

$$\|f\|_{\dot{B}^{\alpha}_{p,\max\{p,q\}}} \leqslant \|f\|_{\dot{F}^{\alpha}_{p,q}} \leqslant \|f\|_{\dot{B}^{\alpha}_{p,\min\{p,q\}}},$$

and, if $0 < q_1 < q_2 \leqslant \infty$,

$$\|f\|_{\dot{B}^{\alpha}_{p,q_2}} \leqslant \|f\|_{\dot{B}^{\alpha}_{p,q_1}}.$$

Moreover,

$$\|f\|_{\dot{B}^{\alpha-n/p}_{\infty,q}} \leqslant C\|f\|_{\dot{B}^{\alpha}_{p,q}}.$$

(ii) If $\alpha > n/p$ then $f \in \dot{B}^{\alpha}_{p,q}$ implies that f is a slowly increasing, continuous function.

PROOF. We refer the reader to [**2**] for the proof of (i). To prove (ii) suppose $f \in \dot{B}^{\alpha}_{p,q}$ and $\alpha > n/p$. We begin by defining, for $|\xi| > 0$,

$$\widehat{\phi}(\xi) = \sum_{j=-\infty}^{0} \widehat{\varphi}(2^{-j}\xi)\widehat{\varphi}(2^{-j}\xi)$$

where, by our convention, $\varphi \in \mathcal{S}$ satisfies (5) and (6). If we take $\widehat{\phi}(0) = 1$ then ϕ satisfies

$$\widehat{\phi}(\xi) = \begin{cases} 1 & |\xi| \leqslant 1 \\ 0 & |\xi| \geqslant 2 \end{cases}$$

and hence $\phi \in \mathcal{S}$. Moreover, for $m, M \in \mathbb{Z}$ with $M > m$,

$$\begin{aligned} \sum_{j=m+1}^{M} \widehat{\varphi}(2^{-j}\xi)\widehat{\varphi}(2^{-j}\xi) &= \sum_{j=-\infty}^{M} \widehat{\varphi}(2^{-j}\xi)\widehat{\varphi}(2^{-j}\xi) - \sum_{j=-\infty}^{m} \widehat{\varphi}(2^{-j}\xi)\widehat{\varphi}(2^{-j}\xi) \\ &= \widehat{\phi}(2^{-M}\xi) - \widehat{\phi}(2^{-m}\xi) \end{aligned}$$

and hence

$$\sum_{j=m+1}^{M} \varphi_j * \varphi_j(x) = \phi_M(x) - \phi_m(x). \tag{7}$$

Take $\mu \in \mathcal{S}$ in such that

$$\widehat{\mu}(\xi) = \begin{cases} 1 & |\xi| \leqslant 2 \\ 0 & |\xi| \geqslant 3 \end{cases}$$

and let $g = f - \mu * f$. Since $\widehat{\mu} = 1$ on the support of $\widehat{\phi}$ we have, for any $\xi \in \mathbb{R}^n$, $\widehat{\phi}(\xi)\widehat{\mu}(\xi) = \widehat{\phi}(\xi)$. Therefore

$$\phi * g = \phi * f - \phi * \eta * f = \phi * f - \phi * f = 0. \tag{8}$$

Similarly, as supp $\widehat{\varphi} = \{1/2 \leqslant |\xi| \leqslant 2\}$, we have $\varphi_j * \mu = 0$ whenever $j \geqslant 3$. Thus

$$\varphi_j * g = \varphi_j * f \tag{9}$$

for any $j \geqslant 3$. Combining (8) and (9) with (7) we obtain, for any $3 \leqslant m < M$,

$$\phi_M * g - \phi_m * g = \sum_{j=m+1}^{M} \varphi_j * \varphi_j * f.$$

Therefore, using the assumption $f \in \dot{B}^{\alpha}_{p,q}$ together with (i), we see that

$$\begin{aligned} \|\phi_M * g - \phi_m * g\|_\infty &\leqslant C \sum_{j=m+1}^{M} \|\varphi_j * f\|_\infty \\ &\leqslant C \sum_{j=m+1}^{M} 2^{-j(\alpha - n/p)} \end{aligned}$$

and hence the family of functions $(\phi_M * g)_{M \in \mathbb{N}}$ forms a Cauchy sequence in L^∞. Thus, as L^∞ forms a Banach space, there exists an $h \in L^\infty$ such that $\phi_M * g$ converges to h uniformly. Moreover, the uniform convergence implies that h is also continuous. Finally, since $\phi_M * g$ converges to g in $\mathcal{S}'$, we must have $g = h \in L^\infty$ and therefore, as $\mu * f$ is a slowly increasing, smooth function,

$$f = g + \mu * f$$

is a slowly increasing, continuous function.

□

The final property of the Besov-Lipschitz and Triebel-Lizorkin spaces we will require is the following characterisation obtained in [**4**] and [**5**]. The methods used to prove this theorem, in particular the use of the Calderón representation formula combined with estimates on the convolution, are similar to the techniques used to prove the characterisations obtained in later chapters.

Before we state the theorem we briefly define the Peetre maximal function, $\mu_k^* f$, by

$$\mu_k^* f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\mu_k * f(y)|}{(1 + 2^k|x - y|)^\lambda}$$

where $\lambda > 0$ is some fixed constant.

Theorem 1.3. *Fix $0 < p, q \leqslant \infty$ and $\alpha \in \mathbb{R}$. Suppose $\mu \in \mathcal{S}$ has $[\alpha]$ vanishing moments (see (2)) and $\eta \in \mathcal{S}$ satisfies the Tauberian condition (see (3)). Then if $\lambda > n/p$ and $f \in \mathcal{S}'$ there exists a polynomial ρ, depending only on f, such that*

$$\left(\int_0^\infty \left(t^{-\alpha} \|\mu_t * (f - \rho)\|_{H^p}\right)^q \frac{dt}{t} \right)^{1/q} \leqslant C_1 \|f\|_{\dot{B}^\alpha_{p,q}} \leqslant C_2 \left(\int_0^\infty \left(t^{-\alpha} \|\eta_t * f\|_p\right)^q \frac{dt}{t} \right)^{1/q}. \tag{10}$$

Similarly, if $p < \infty$ and $\lambda > \max\{n/p, n/q\}$, then for any $f \in \mathcal{S}'$ there exists a polynomial ρ depending only on f such that

$$\left\| \left(\int_0^\infty \left(t^{-\alpha} \mu_t^*(f - \rho)\right)^q \frac{dt}{t} \right)^{1/q} \right\|_p \leqslant C_1 \|f\|_{\dot{F}^\alpha_{p,q}} \leqslant C_2 \left\| \left(\int_0^\infty \left(t^{-\alpha} |\eta_t * f|\right)^q \frac{dt}{t} \right)^{1/q} \right\|_p. \tag{11}$$

Moreover the discrete version of both inequalities also holds.

Proof. See [**4**] and [**5**]. □

We make the following remarks. Firstly, the polynomial term in (10) and (11) cannot be removed. To see this note that the infinite vanishing moments of φ implies that the convolution $\varphi_j * f$ annihilates all polynomials, i.e. $\varphi * \rho = 0$ for any polynomial ρ. On the other hand, as we only require μ to have $[\alpha]$ vanishing moments, the convolution $\mu * \rho$ will not vanish for polynomials with degree higher than $[\alpha]$. Thus taking $f = \rho$ with ρ some polynomial of degree higher than $[\alpha]$ gives $\|f\|_{\dot{B}^\alpha_{p,q}} = 0$ but

$$\left(\int_0^\infty \left(t^{-\alpha} \|\mu_t * f\|_{H^p}\right)^q \frac{dt}{t} \right)^{1/q} > 0.$$

Thus the polynomial term in (10) cannot be removed. A similar comment applies to the Triebel-Lizorkin version, (11).

Secondly, since $\|\mu_t * f\|_p \leqslant \|\mu_t * f\|_{H^p}$ for all $0 < p, t < \infty$ and any $\mu \in \mathcal{S}$, the above theorem implies that

$$\left(\int_0^\infty \left(t^{-\alpha}\|\mu_t * (f-\rho)\|_p\right)^q \frac{dt}{t}\right)^{1/q} \leqslant C\|f\|_{\dot{B}^\alpha_{p,q}},$$

assuming $f \in \mathcal{S}'$ and μ has $[\alpha]$ vanishing moments. In particular, when $p = q = \infty$, we have

$$\sup_{t>0} t^{-\alpha}\|\mu_t * (f-\rho)\|_\infty \leqslant C\|f\|_{\dot{B}^\alpha_{\infty,\infty}}. \tag{12}$$

Moreover, as remarked in [**5**], we may replace $\|\mu_t * (f-\rho)\|_{H^p}$ with $\|\mu_t^*(f-\rho)\|_p$ in (10).

3. Maximal Inequalities

Maximal functions have proven to be an invaluable tool in many areas of mathematics, particularly in Harmonic Analysis. They will also prove crucial in our study as they will allow us to obtain estimates on the convolution without resorting to tools such as Young's Inequality. This will allow us to extend our characterisations to the case $p < 1$, where Young's Inequality no longer holds.

We define the Hardy-Littlewood maximal function $M(f)$ for a measurable function f by

$$M(f)(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{|x-y|\leqslant r} |f(y)|dy$$

where $m(B(x,r))$ is the Lebesgue measure of the ball centered at x and of radius r. The main result we will use is following vector-valued maximal inequality of Fefferman and Stein.

Theorem 1.4. *Fix $1 < p, q \leqslant \infty$, $p < \infty$. Then for any sequence of measurable functions $(f_k)_{k\in\mathbb{Z}}$ we have*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} M(f_k)^q\Big)^{1/q}\Big\|_p \leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}} |f_k|^q\Big)^{1/q}\Big\|_p.$$

Proof. The case $p, q < \infty$ was proven in [**7**]. The case $q = \infty$ is a consequence of the usual scalar-valued case as

$$\sup_{k\in\mathbb{Z}} M(f_k)(x) \leqslant M(\sup_{k\in\mathbb{Z}} |f_k|)(x).$$

□

We will also need the following elementary inequality.

PROPOSITION 1.5. *Fix $b > n$ and suppose g is a measurable function. Then, for any $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$,*

$$\int_{\mathbb{R}^n} \frac{|g(y)|}{(1+2^j|x-y|)^b}\, 2^{jn} dy \leqslant CM(g)(x).$$

PROOF. For any measurable function g and $b > 0$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|g(y)|}{(1+2^j|x-y|)^b}\, 2^{jn} dy &= \int_{|x-y|<2^{-j}} \frac{|g(y)|}{(1+2^j|x-y|)^b} 2^{jn} dy \\ &\qquad + \sum_{k=0}^{\infty} \int_{2^{k-j} \leqslant |x-y| < 2^{k+1-j}} \frac{|g(y)|}{(1+2^j|x-y|)^b}\, 2^{jn} dy \\ &\leqslant 2^{jn} \int_{|x-y|<2^{-j}} |g(y)|\, dy \\ &\qquad + \sum_{k=0}^{\infty} 2^{-k(b-n)+n} 2^{-(k+1-j)n} \int_{|x-y| \leqslant 2^{k+1-j}} |g(y)|\, dy \\ &\leqslant CM(g)(x) + CM(g)(x) \sum_{k=0}^{\infty} 2^{-k(b-n)}\end{aligned}$$

and hence if $b > n$ we obtain

$$\int_{\mathbb{R}^n} \frac{|g(y)|}{(1+2^j|x-y|)^b} 2^{jn} dy \leqslant CM(g)(x).$$

□

4. Bounded Distributions

Recall that for $\eta \in \mathcal{S}$ and $f \in \mathcal{S}'$, the convolution $\eta * f$ can be defined as a function by letting

$$\eta * f(x) = f(\tau_x \widetilde{\eta}),$$

where, as previously, we define $\widetilde{\eta}(y) = \eta(-y)$ and $\tau_x \eta(y) = \eta(y-x)$. Moreover the function $\eta * f$ is infinitely differentiable, slowly increasing, and for any $\phi \in \mathcal{S}$ we have

$$(\eta * f)(\phi) = f(\widetilde{\eta} * \phi).$$

Our aim is to define the convolution $\eta * f$ without the assumption $\eta \in \mathcal{S}$. This is possible provided we restrict the class of distributions under consideration.

Following E. Stein [**12**, pg 89] we say that a (tempered) distribution $f \in \mathcal{S}'$ is a *bounded distribution* if, for every $\phi \in \mathcal{S}$,

$$\phi * f \in L^{\infty}. \tag{13}$$

In this case we can then proceed to define the convolution $\psi * f$ as a distribution for any $\psi \in L^1$ by letting

$$\psi * f(\phi) = \int_{\mathbb{R}^n} \widetilde{\phi} * f(x)\, \widetilde{\psi}(x)\, dx.$$

(Note that this integral is absolutely convergent since we have assumed $\widetilde{\phi} * f \in L^\infty$). From this definition it is easy enough to show that $\psi * f$ is also a bounded distribution and we have, for any $\psi, \eta \in L^1$,

$$\psi * (\eta * f) = \eta * (\psi * f)$$

in the sense of distributions.

Now suppose we have $f \in \mathcal{S}'$ such that $D^\kappa f$ is a bounded distribution for any $|\kappa| = k$, where $k \in \mathbb{N}$. This assumption then implies that for any $\phi \in \mathcal{S}$ we have

$$|\phi * f(x)| \leqslant C(1 + |x|)^k$$

(see Lemma B.4). Thus if we take ψ such that $x^\kappa \psi \in L^1$ for each $|\kappa| \leqslant k$, we see that we can define the convolution $\psi * f$ as a distribution as before by taking

$$(\psi * f)(\phi) = \int_{\mathbb{R}^n} \widetilde{\phi} * f(x)\, \widetilde{\psi}(x)\, dx. \tag{14}$$

We prove the following elementary results regarding this definition.

Lemma 1.6. *Take $f \in \mathcal{S}'$ such that $D^\kappa f$ is a bounded distribution for every $|\kappa| = k$ and suppose $(1 + |\cdot|)^k \psi(\cdot) \in L^1$. Then for any $\phi \in \mathcal{S}$*

$$\phi * \big(\psi * f\big)(x) = \psi * \big(\phi * f\big)(x). \tag{15}$$

Proof. The proof proceeds by unpacking the definitions of the convolutions $\psi * f$ and $\phi * (\psi * f)$. Assume f and ψ satisfy conditions of the lemma. By the definition of the convolution, (14), we have, for any $\phi \in \mathcal{S}$,

$$\begin{aligned}\phi * \big(\psi * f\big)(x) &= \big(\psi * f\big)(\tau_x \widetilde{\phi}) \\ &= \int_{\mathbb{R}^n} \widetilde{\tau_x \widetilde{\phi}} * f(y)\, \widetilde{\psi}(y) dy.\end{aligned}$$

Again by the definition of the convolution we see that

$$\begin{aligned}\widetilde{\tau_x \widetilde{\phi}} * f(y) &= f\big(\tau_y \tau_x \widetilde{\phi}\big) \\ &= \phi * f(x + y).\end{aligned}$$

Therefore

$$\begin{aligned}\phi * \big(\psi * f\big)(x) &= \int_{\mathbb{R}^n} \phi * f(x+y)\widetilde{\psi}(y)dy \\ &= \psi * \big(\phi * f\big)(x).\end{aligned}$$

□

The next result shows that the convolution defined as above remains commutative.

LEMMA 1.7. *Take $f \in \mathcal{S}'$ such that $D^\kappa f$ is a bounded distribution for every $|\kappa| = k$ and suppose $(1+|\cdot|)^k\psi(\cdot)$, $(1+|\cdot|)^k\eta(\cdot) \in L^1$. Then, for any $|\kappa| = k$, $D^\kappa\big(\psi * f\big)$ is a bounded distribution and moreover*

$$\eta * \big(\psi * f\big) = \psi * \big(\eta * f\big) = (\psi * \eta) * f. \tag{16}$$

PROOF. An application of Fubini's Theorem together with the elementary inequality

$$(1+|x|)^k \leqslant (1+|x-y|)^k(1+|y|)^k$$

gives

$$\begin{aligned}\int_{\mathbb{R}^n} |\psi * \eta(x)|(1+|x|)^k\,dx &\leqslant \int_{\mathbb{R}^n} |\psi(y)|(1+|y|)^k \int_{\mathbb{R}^n} |\eta(x-y)|(1+|x-y|)^k\,dx\,dy \\ &\leqslant C.\end{aligned}$$

Hence $(1+|\cdot|)^k(\psi * \eta)(\cdot) \in L^1$ and so, by using (14), we can define $(\psi * \eta) * f$ as a distribution. On the other hand the previous lemma shows that, for any $\phi \in \mathcal{S}$ and $|\kappa| = k$,

$$\begin{aligned}\phi * D^\kappa\big(\psi * f\big)(x) &= D^\kappa\phi * \big(\psi * f\big)(x) \\ &= \psi * \big(D^\kappa\phi * f\big)(x).\end{aligned}$$

and hence $\phi * D^\kappa\big(\psi * f\big) \in L^\infty$. Thus by definition we see that $D^\kappa\big(\psi * f\big)$ is a bounded distribution for every $|\kappa| = k$. Therefore each of the convolutions in (16) is a well-defined distribution.

Take any $\phi \in \mathcal{S}$. Using the previous lemma it is clear that, for any $x \in \mathbb{R}^n$,

$$\phi * \Big(\psi * \big(\eta * f\big)\Big)(x) = \phi * \Big(\eta * \big(\psi * f\big)\Big)(x) = \phi * \Big((\psi * \eta) * f\Big)(x).$$

In particular, for $x = 0$ we have

$$\psi * \big(\eta * f\big)(\phi) = \widetilde{\phi} * \Big(\psi * \big(\eta * f\big)\Big)(0) = \widetilde{\phi} * \Big(\eta * \big(\psi * f\big)\Big)(0) = \eta * \big(\psi * f\big)(\phi)$$

and similarly

$$(\psi * \eta) * f(\phi) = \widetilde{\phi} * \Big((\psi * \eta) * f\Big)(0) = \widetilde{\phi} * \Big(\eta * \big(\psi * f\big)\Big)(0) = \eta * \big(\psi * f\big)(\phi).$$

□

Note that if we take ψ and f as in the above lemma and suppose that $\eta, \phi \in \mathcal{S}$ then we have

$$\eta * (\psi * f)(\phi) = (\eta * \psi) * f(\phi).$$

Thus the distribution $(\eta * \psi) * f$ is a function and moreover we see that

$$(\eta * \psi) * f(x) = \psi * (\eta * f)(x) = \eta * (\psi * f)(x).$$

Therefore we may omit the brackets when writing multiple convolutions such as $\eta * \psi * f(x)$.

We present one more result in this section.

PROPOSITION 1.8. *Suppose $f \in \mathcal{S}'$ and $D^\kappa f$ is a bounded distribution for every $|\kappa| = k+1$. If $\phi \in \mathcal{S}$ has k vanishing moments, then the smooth function $\phi * f$ is bounded.*

PROOF. By Theorem 1.1, for each $|\kappa| = k+1$ there exists $\phi^\kappa \in \mathcal{S}$ such that

$$\phi = \sum_{|\kappa|=k+1} D^\kappa \phi^\kappa.$$

Thus, as $D^\kappa f$ is a bounded distribution for every $|\kappa| = k+1$, we have

$$|\phi * f(x)| \leqslant \sum_{|\kappa|=k+1} |\phi^\kappa * D^\kappa f(x)| \leqslant C.$$

□

CHAPTER 2

Necessary Conditions

In this chapter necessary conditions for a distribution to belong to either the Triebel-Lizorkin space, $\dot{F}^{\alpha}_{p,q}$, or the Besov-Lipschitz space, $\dot{B}^{\alpha}_{p,q}$, will be considered. We aim to prove the following theorem. Recall that the Peetre maximal function, $\psi_t^* f(x)$, is defined by

$$\psi_t^* f(x) = \sup_{y\in\mathbb{R}^n} \frac{|\psi_t * f(y)|}{(1+|x-y|/t)^{\lambda}}.$$

THEOREM 2.1. *Let $0 < p, q \leqslant \infty$, $\alpha, \lambda \in \mathbb{R}$ and choose $m \in \mathbb{N}$ such that $m > \lambda - \alpha$. Suppose $(1+|\cdot|)^{\lambda}\psi(\cdot) \in L^1$ satisfies:*

(i) The kernel ψ has $[\alpha]$ vanishing moments and $(1+|\cdot|)^{[\alpha]+\lambda+1}\psi(\cdot) \in L^1$.
(ii) We have $\psi \in C^m$ and for each $|\kappa| = m$,

$$(1+|\cdot|)^{\lambda} D^{\kappa}\psi(\cdot) \in L^1.$$

If $\lambda > n/p$, then for any $f \in \dot{B}^{\alpha}_{p,q}$ there exists a polynomial ρ such that

$$\Big(\sum_{j\in\mathbb{Z}} \Big(2^{\alpha j}\|\psi_j^*(f-\rho)\|_p\Big)^q\Big)^{1/q} \leqslant C\|f\|_{\dot{B}^{\alpha}_{p,q}}.$$

Similarly, if $p < \infty$ and $\lambda > \max\{n/p, n/q\}$, we have for any $f \in \dot{F}^{\alpha}_{p,q}$ a polynomial ρ such that

$$\Big\|\Big(\sum_{j\in\mathbb{Z}} (2^{j\alpha}|\psi_j^*(f-\rho)|)^q\Big)^{1/q}\Big\|_p \leqslant C\|f\|_{\dot{F}^{\alpha}_{p,q}}.$$

Note that we no longer require the kernel ψ to belong to $\mathcal{S}$ and so the first obstacle is to define the convolution $\psi * f$. We proceed by first showing that if $f \in \dot{B}^{\alpha}_{p,q}$ or $f \in \dot{F}^{\alpha}_{p,q}$ and $\phi \in \mathcal{S}$ has $[\alpha]$ vanishing moments, then the convolution $\phi_t * (f-\rho)$ satisfies a certain estimate, where ρ is some polynomial. This gives a connection between bounded distributions and the spaces $\dot{F}^{\alpha}_{p,q}$ and $\dot{B}^{\alpha}_{p,q}$, allowing us to define $\psi*(f-\rho)$ as a distribution. However, for the inequalities in Theorem 2.1 to make sense, we will need to show $\psi*(f-\rho)$ is a function. Remarkably, by using the Calderón reproducing formula, we show that this is indeed the case and in fact the convolution $\psi * (f-\rho)$ forms a well-defined, bounded, continuous function; see Theorem 2.5.

1. The Calderón Reproducing Formula

The Calderón reproducing formula will form an essential part in the proof of Theorem 2.1. This formula will allow us to reconstruct a function g from the sequence of convolutions $(\varphi_j * \varphi_j * g)_{j\in\mathbb{Z}}$. More precisely, the Calderón reproducing formula states that we have

$$g = \sum_{j\in\mathbb{Z}} \varphi_j * \varphi_j * g \tag{17}$$

in some appropriate sense and for some suitable g. The proof of (17) will depend on the equality

$$\sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^{-j}\xi)\widehat{\varphi}(2^{-j}\xi) = 1 \tag{18}$$

which, by our assumption, holds for every $\xi \in \mathbb{R}^n \setminus \{0\}$. Intuitively, we can deduce (18) from (17) by multiplying by $\widehat{g}$ and taking the inverse Fourier transform to obtain

$$g = \sum_{j\in\mathbb{Z}} \varphi_j * \varphi_j * g.$$

The major difficulty in the proof of (17) is making this formal computation rigorous.

We begin by proving the following lemma which gives a useful estimate on the members of $\dot{B}^{\alpha}_{p,q}$ and $\dot{F}^{\alpha}_{p,q}$.

Lemma 2.2. *Suppose $f \in \dot{B}^{\alpha}_{p,q}$ or $\dot{F}^{\alpha}_{p,q}$. If $\mu \in \mathcal{S}$ has $[\alpha - n/p]$ vanishing moments, then there exists a polynomial ρ such that*

$$\|\mu_t * (f-\rho)\|_\infty \leqslant Ct^{\alpha-n/p}$$

for any $0 < t < \infty$.

Proof. From Theorem 1.2 we have $\dot{B}^{\alpha}_{p,q} \cup \dot{F}^{\alpha}_{p,q} \subseteq \dot{B}^{\alpha}_{p,\infty}$ and $\dot{B}^{\alpha}_{p,\infty} \subseteq \dot{B}^{\alpha-n/p}_{\infty,\infty}$. Thus the required estimate follows from (12) after Theorem 1.3.

□

Lemma 2.2 then gives the following corollary.

corollary 2.3. *Suppose $f \in \dot{B}^{\alpha}_{p,q}$ or $\dot{F}^{\alpha}_{p,q}$. Then there exists a polynomial ρ such that $D^\kappa(f-\rho)$ is a bounded distribution for any $|\kappa| > [\alpha - n/p]$.*

Proof. Take $\mu \in \mathcal{S}$ and suppose $f \in \dot{F}^{\alpha}_{p,q}$ or $\dot{B}^{\alpha}_{p,q}$. If $|\kappa| > [\alpha - n/p]$ then $D^\kappa\mu$ has $[\alpha - n/p]$ vanishing moments and hence, taking $t = 1$ in Lemma 2.2 we have

$$\|\mu * D^\kappa(f-\rho)\|_\infty = \|D^\kappa\mu * (f-\rho)\|_\infty \leqslant C.$$

Therefore $D^\kappa(f-\rho)$ is a bounded distribution. $\square$

Corollary 2.3 implies that for any $f \in \dot{B}^\alpha_{p,q}$ or $\dot{F}^\alpha_{p,q}$ there exists a polynomial ρ such that the convolution $\psi*(f-\rho)$ can be defined as a distribution as long as $(1+|\cdot|)^{[\alpha-n/p]+1}\psi(\cdot) \in L^1$. Where, as in (14) of Chapter 2, Section 4, we take

$$\psi * (f-\rho)(\mu) = \int_{\mathbb{R}^n} \widetilde{\mu} * (f-\rho)(x)\, \widetilde{\psi}(x)\, dx.$$

Before we come to the Calderón reproducing formula (which shows that under certain conditions on ψ and f the convolution $\psi * f$ is a bounded continuous function) we will present the following technical lemma giving us control over $\eta_s * \mu_t$ in terms of the dilation factors t and s. This lemma was proven in [**4**] with the assumption $\eta, \mu \in \mathcal{S}$ but it was noted there that it actually holds under weaker assumptions. We include a proof of this slightly more general version for convenience.

LEMMA 2.4. *Take $k \in \mathbb{N}$ and $\lambda \geqslant 0$. Assume η has k vanishing moments and satisfies*

$$(1+|\cdot|)^{\lambda+k+1}\eta(\cdot) \in L^1$$

and $\mu \in C^{k+1}$ with

$$(1+|\cdot|)^{\lambda} D^\kappa \mu(\cdot) \in L^1$$

for every $|\kappa| = k+1$. Then for any $0 < s \leqslant t < \infty$ we have

$$\int_{\mathbb{R}^n} |\eta_s * \mu_t(x)| \left(1+\frac{|x|}{t}\right)^{\lambda} dx \leqslant C\left(\frac{s}{t}\right)^{k+1}.$$

PROOF. Assume η and μ satisfy the above conditions. Using Taylor's formula together with the moment condition on η gives, for each $x \in \mathbb{R}^n$,

$$\begin{aligned}
|\eta_s * \mu_t(x)| &= \Big| \int_{\mathbb{R}^n} \eta_s(ty)\, \mu\Big(\frac{x}{t} - y\Big)\, dy \Big| \\
&\leqslant \sum_{|\kappa|=k+1} c_\kappa \int_{\mathbb{R}^n} |y|^{k+1} |\eta_s(ty)| \int_0^1 \rho^k \Big| D^\kappa \mu\Big(\frac{x}{t} - \rho y\Big)\Big| d\rho\, dy \\
&= \sum_{|\kappa|=k+1} c_\kappa I_\kappa(x).
\end{aligned}$$

It remains to prove that for each $|\kappa| = k+1$ we have

$$\int_{\mathbb{R}^n} I_\kappa(x) \left(1+\frac{|x|}{t}\right)^{\lambda} dx \leqslant C\left(\frac{s}{t}\right)^{k+1}.$$

To this end, by noting the elementary inequality $(1+|x|)^\lambda \leqslant (1+|x-\rho y|)^\lambda (1+\rho|y|)^\lambda$ and using the decay on $D^\kappa \mu$, we obtain

$$\begin{aligned} (19) \quad \int_{\mathbb{R}^n} \Big(1+\frac{|x|}{t}\Big)^\lambda \int_0^1 \rho^k \Big| D^\kappa \mu\Big(\frac{x}{t}-\rho y\Big)\Big| d\rho\, dx \\ &\leqslant \int_0^1 \rho^k (1+\rho|y|)^\lambda\, t^n \int_{\mathbb{R}^n} (1+|x-\rho y|)^\lambda |D^\kappa \mu(x-\rho y)| dx\, d\rho \\ &\leqslant Ct^n(1+|y|)^\lambda. \end{aligned}$$

Thus combining (19) with an application of Fubini's Theorem and using the assumption $0 < s \leqslant t < \infty$ shows

$$\begin{aligned} \int_{\mathbb{R}^n} I_\kappa(x)\Big(1+\frac{|x|}{t}\Big)^\lambda dx &\leqslant \int_{\mathbb{R}^n} |y|^{k+1} |\eta_s(ty)| \int_{\mathbb{R}^n} \Big(1+\frac{|x|}{t}\Big)^\lambda \int_0^1 \rho^k \Big| D^\kappa \mu\Big(\frac{x}{t}-\rho y\Big)\Big| d\rho\, dx\, dy \\ &\leqslant C \int_{\mathbb{R}^n} t^n |y|^{k+1} |\eta_s(ty)| (1+|y|)^\lambda dy \\ &\leqslant C\Big(\frac{s}{t}\Big)^{k+1} \int_{\mathbb{R}^n} \Big|\frac{t}{s}y\Big|^{k+1} |\eta_{s/t}(y)| \Big(1+\Big|\frac{t}{s}y\Big|\Big)^\lambda dy \\ &= C\Big(\frac{s}{t}\Big)^{k+1}. \end{aligned}$$

□

We now prove the following version of the Calderón reproducing formula. Recall that by our convention $\varphi \in \mathcal{S}$ satisfies, for every $|\xi| > 0$,

$$\sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^{-j}\xi)\widehat{\varphi}(2^{-j}\xi) = 1$$

and moreover supp $\widehat{\varphi} \subseteq \{1/2 \leqslant |\xi| \leqslant 2\}$.

Theorem 2.5. *Let $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}$ with $m > -\alpha$. Take $f \in \mathcal{S}'$ such that*

$$\|\eta_t * f\|_\infty \leqslant Ct^\alpha \tag{20}$$

for any $\eta \in \mathcal{S}$ with $[\alpha]$ vanishing moments. Suppose $\psi \in L^1$ satisfies:

(i) The kernel ψ has $[\alpha]$ vanishing moments and $(1+|\cdot|)^{[\alpha]+1}\psi(\cdot) \in L^1$.
(ii) We have $\psi \in C^m$ and for each $|\kappa| = m$, $D^\kappa \psi \in L^1$.

Then the distribution $\psi * f$ is a bounded, continuous function such that, for every $t > 0$,

$$\|\psi_t * f\|_\infty \leqslant Ct^\alpha. \tag{21}$$

Moreover for any $x \in \mathbb{R}^n$ we have

$$\psi * f(x) = \sum_{j\in\mathbb{Z}} \psi * \varphi_j * \varphi_j * f(x) \tag{22}$$

where the series converges uniformly.

PROOF. We begin by noting that the condition (i) on ψ together with the fact that f satisfies (20), implies that we can define $\psi * f$ as a distribution. To prove that $\psi * f$ is a function we will use the Calderón reproducing formula together with a limiting argument.

For $\xi \neq 0$ let

$$\widehat{\phi}(\xi) = \sum_{j=-\infty}^{0} \widehat{\varphi}(2^{-j}\xi)\widehat{\varphi}(2^{-j}\xi) \tag{23}$$

and take $\widehat{\phi}(0) = 1$. We briefly note that (23) is a finite sum away from the origin and $\phi \in \mathcal{S}$. Define

$$\widehat{I}_{N,M} = \sum_{j=N+1}^{M} \widehat{\varphi}(2^{-j}\xi)\widehat{\varphi}(2^{-j}\xi) = \widehat{\phi}(2^{-M}\xi) - \widehat{\phi}(2^{-N}\xi).$$

Taking the convolution of $I_{N,M} \in \mathcal{S}$ with the distribution $\psi * f$ we obtain

$$I_{N,M} * \psi * f(x) = \sum_{j=N+1}^{M} \varphi_j * \varphi_j * (\psi * f)(x) = \phi_M * (\psi * f)(x) - \phi_N * (\psi * f)(x). \tag{24}$$

Since $\phi \in \mathcal{S}$, Lemma 1.6 gives $\phi_N * (\psi * f)(x) = \psi * (\phi_N * f)$. Thus by expanding the smooth function $\phi_N * f$ using Taylor's Theorem and applying the moment condition on ψ we see that

$$|\phi_N * \big(\psi * f\big)(x)| \leqslant \sum_{|\kappa|=[\alpha]+1} c_\kappa \int_{\mathbb{R}^n} |\psi(y)|\, |y|^{[\alpha]+1} \int_0^1 \rho^{[\alpha]} |D^\kappa\big(\phi_N * f\big)(x-\rho y)|\, d\rho\, dy.$$

Now as $D^\kappa\phi \in \mathcal{S}$ has $[\alpha]$ vanishing moments (since $|\kappa| = [\alpha]+1$) we have, for any $z \in \mathbb{R}^n$ and each $|\kappa| = [\alpha]+1$,

$$|D^\kappa\big(\phi_N * f\big)(z)| = 2^{N([\alpha]+1)} |\big(D^\kappa\phi\big)_N * f(z)| \leqslant C 2^{N([\alpha]+1-\alpha)}$$

and hence

$$\begin{aligned}|\phi_N * \big(\psi * f\big)(x)| &\leqslant \sum_{|\kappa|=[\alpha]+1} c_\kappa \int_{\mathbb{R}^n} |\psi(y)|\, |y|^{[\alpha]+1} \int_0^1 \rho^{[\alpha]} |D^\kappa\big(\phi_N * f\big)(x-\rho y)|\, d\rho\, dy \\ &\leqslant C 2^{N([\alpha]+1-\alpha)} \int_{\mathbb{R}^n} |\psi(y)|\, |y|^{[\alpha]+1}\, dy \\ &\leqslant C 2^{N([\alpha]+1-\alpha)}.\end{aligned}$$

Therefore, letting N tend to $-\infty$, we obtain

$$\lim_{N\to-\infty} \|\phi_N * (\psi * f)\|_\infty = 0. \tag{25}$$

Combining (25) with (24) then gives, for every $x \in \mathbb{R}^n$ and $M \in \mathbb{N}$,

$$\sum_{j=-\infty}^{M} \psi * \varphi_j * \varphi_j * f(x) = \phi_M * \big(\psi * f\big)(x).$$

Since $\int_{\mathbb{R}^n} \phi = 1$, the family of functions $(\phi_M)_{M\in\mathbb{N}}$ forms an approximation to the identity, which implies that $\lim_{M\to\infty} \phi_M * \big(\psi * f\big) = \psi * f$ in $\mathcal{S}'$. Thus if we can prove that

$$\sum_{j\in\mathbb{Z}} \|\varphi_j * \varphi_j * \psi_t * f\|_\infty \leqslant Ct^\alpha \tag{26}$$

then firstly the required estimate, (21), holds and secondly, as the summation $\sum_{j\in\mathbb{Z}} \psi * \varphi_j * \varphi_j * f(x)$ converges uniformly, we see that the distribution $\psi * f$ is the uniform limit of continuous functions and hence continuous. Finally this would imply that we have the pointwise equality

$$\psi_t * f(x) = \sum_{j\in\mathbb{Z}} \psi_t * \varphi_j * \varphi_j * f(x).$$

Fix $t > 0$ and choose $k \in \mathbb{Z}$ such that $2^{-k} \leqslant t < 2^{-k+1}$. Then using Lemma 2.4 we have, for $j < k$,

$$\|\psi_t * \varphi_j\|_1 \leqslant C\Big(\frac{t}{2^{-j}}\Big)^{[\alpha]+1} \leqslant C2^{(j-k)([\alpha]+1)}.$$

Similarly, since φ has infinitely vanishing moments, for $j \geqslant k$ we have

$$\|\psi_t * \varphi_j\|_1 \leqslant C\Big(\frac{2^{-j}}{t}\Big)^{m} \leqslant C2^{(k-j)m}.$$

Therefore, combining these estimates and using the assumption (20), we obtain

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \|\varphi_j * \varphi_j * \psi_t * f\|_\infty &\leqslant \sum_{j\in\mathbb{Z}} \|\varphi_j * \psi_t\|_1 \|\varphi_j * f\|_\infty \\
&= \sum_{j\geqslant k} \|\varphi_j * \psi_t\|_1 \|\varphi_j * f\|_\infty + \sum_{j<k} \|\varphi_j * \psi_t\|_1 \|\varphi_j * f\|_\infty \\
&\leqslant C2^{-k\alpha} \sum_{j\geqslant k} 2^{(k-j)(m+\alpha)} + C2^{-k\alpha} \sum_{j<k} 2^{(j-k)([\alpha]+1-\alpha)} \\
&\leqslant C2^{-k\alpha} \\
&\leqslant Ct^\alpha.
\end{aligned}$$

□

Note that by our definition of the moment condition, the assumption (i) on ψ becomes void if $\alpha < 0$. Similarly, by a careful examination of the above proof, we see that the condition (ii) becomes void when $\alpha > 0$.

Theorem 2.5 together with Lemma 2.2 shows that for any $f \in \dot{B}^{\alpha}_{p,q}$ or $\dot{F}^{\alpha}_{p,q}$, as long as ψ satisfies (i) and (ii), there exists a polynomial ρ such that the convolution $\psi * (f-\rho)$ is a bounded continuous function. Therefore we have shown that the statement of Theorem 2.1 makes sense.

2. Proof of Theorem 2.1

To simplify the proof of Theorem 2.1 we use the following consequence of Lemma 2.4.

LEMMA 2.6. *Take $m, M \in \mathbb{N}$, $\lambda \geqslant 0$, and suppose $m > \lambda$. Assume $\psi \in C^m$ has M vanishing moments and satisfies*

$$(1+|\cdot|)^{\lambda+M+1}\psi(\cdot) \in L^1$$

and for every $|\kappa| = m$

$$(1+|\cdot|)^{\lambda} D^{\kappa}\psi(\cdot) \in L^1.$$

Then for any $j, k \in \mathbb{Z}$ and $x, z \in \mathbb{R}^n$ we have

$$\int_{\mathbb{R}^n} |\psi_k * \varphi_j(z-y)|\,(1+2^j|x-y|)^{\lambda} dy \leqslant C \min\{2^{(M+1)(j-k)}, 2^{(m-\lambda)(k-j)}\}\,(1+2^k|x-z|)^{\lambda}$$

PROOF. Suppose the above conditions on ψ hold and recall that $\varphi \in \mathcal{S}$ satisfies supp $\widehat{\varphi} \subseteq \{1/2 \leqslant |\xi| \leqslant 2\}$, which implies φ has infinite vanishing moments. If $k \geqslant j$ then since ψ has M vanishing moments and $(1+|\cdot|)^{\lambda+M+1}\psi(\cdot) \in L^1$ we have, by Lemma 2.4,

$$\int_{\mathbb{R}^n} |\psi_k * \varphi_j(y)|(1+2^j|y|)^{\lambda} dy \leqslant C2^{(M+1)(j-k)}.$$

The elementary inequality $(1+2^j|x-y|)^{\lambda} \leqslant (1+2^k|x-z|)^{\lambda}(1+2^j|z-y|)^{\lambda}$ then gives

$$\begin{aligned}\int_{\mathbb{R}^n} |\psi_k * \varphi_j(z-y)|\,(1+2^j|x-y|)^{\lambda} dy &\leqslant (1+2^k|x-z|)^{\lambda} \int_{\mathbb{R}^n} |\psi_k * \varphi_j(y)|\,(1+2^j|y|)^{\lambda} dy \\ &\leqslant C2^{(M+1)(j-k)}(1+2^k|x-z|)^{\lambda}\end{aligned}$$

as required.

If we now assume $k < j$ then since $\psi \in C^m$ with $(1+|\cdot|)^{\lambda} D^{\kappa}\psi(\cdot) \in L^1$ we can again apply Lemma 2.4 to obtain

$$\int_{\mathbb{R}^n} |\psi_k * \varphi_j(y)|(1+2^k|y|)^{\lambda} dy \leqslant C2^{m(k-j)}.$$

Again using the assumption $k < j$ we have the elementary inequality

$$\begin{aligned}(1+2^j|x-y|)^{\lambda} &\leqslant 2^{\lambda(j-k)}(1+2^k|x-y|)^{\lambda} \\ &\leqslant 2^{\lambda(j-k)}(1+2^k|x-z|)^{\lambda}(1+2^k|z-y|)^{\lambda}.\end{aligned}$$

Therefore, for any $x, z \in \mathbb{R}^n$ and $k < j$,

$$\begin{aligned}\int_{\mathbb{R}^n} |\psi_k * \varphi_j(z-y)|\,(1+2^j|x-y|)^\lambda dy &\leqslant 2^{\lambda(j-k)}(1+2^k|x-z|)^\lambda \int_{\mathbb{R}^n} |\psi_k * \varphi_j(y)|\,(1+2^k|y|)^\lambda dy \\ &\leqslant C2^{(m-\lambda)(k-j)}(1+2^k|x-z|)^\lambda.\end{aligned}$$

□

For the reader's convenience, we restate Theorem 2.1.

THEOREM 2.1. *Let $0 < p, q \leqslant \infty$, $\alpha, \lambda \in \mathbb{R}$ and choose $m \in \mathbb{N}$ such that $m > \lambda - \alpha$. Suppose $(1 + |\cdot|)^\lambda \psi(\cdot) \in L^1$ satisfies:*

(i) The kernel ψ has $[\alpha]$ vanishing moments and $(1+|\cdot|)^{[\alpha]+\lambda+1}\psi(\cdot) \in L^1$.
(ii) We have $\psi \in C^m$ and for each $|\kappa| = m$,

$$(1+|\cdot|)^\lambda D^\kappa \psi(\cdot) \in L^1.$$

If $\lambda > n/p$, then for any $f \in \dot{B}^\alpha_{p,q}$ there exists a polynomial ρ such that

$$\Big(\sum_{j\in\mathbb{Z}} \Big(2^{\alpha j}\|\psi_j^*(f-\rho)\|_p\Big)^q\Big)^{1/q} \leqslant C\|f\|_{\dot{B}^\alpha_{p,q}}.$$

Similarly, if $p < \infty$ and $\lambda > \max\{n/p, n/q\}$, we have for any $f \in \dot{F}^\alpha_{p,q}$ a polynomial ρ such that

$$\Big\|\Big(\sum_{j\in\mathbb{Z}} (2^{j\alpha}|\psi_j^*(f-\rho)|)^q\Big)^{1/q}\Big\|_p \leqslant C\|f\|_{\dot{F}^\alpha_{p,q}}.$$

PROOF. Suppose ψ satisfies the above conditions. If $f \in \dot{B}^\alpha_{p,q}$ or $\dot{F}^\alpha_{p,q}$ then Lemma 2.2 shows there exists a polynomial ρ such that we can apply the Calderón type formula of Theorem 2.5 to obtain

$$\psi_k * (f-\rho)(z) = \sum_{j\in\mathbb{Z}} \varphi_j * \varphi_j * \psi_k * (f-\rho)(z). \tag{27}$$

Let $g = f - \rho$. Combining (27) with Lemma 2.6 we have

$$\begin{aligned}|\psi_k * g(z)| &\leqslant \sum_{j\in\mathbb{Z}} |\psi_k * \varphi_j * \varphi_j * g(z)| \\ &\leqslant \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} |\psi_k * \varphi_j(z-y)|(1+2^j|x-y|)^\lambda \frac{|\varphi_j * g(y)|}{(1+2^j|x-y|)^\lambda}\, dy \\ &\leqslant \sum_{j\in\mathbb{Z}} \varphi_j^* g(x) \int_{\mathbb{R}^n} |\psi_k * \varphi_j(z-y)|(1+2^j|x-y|)^\lambda\, dy \\ &\leqslant C(1+2^k|z-x|)^\lambda \sum_{j\in\mathbb{Z}} a_{k-j}\, 2^{\alpha(j-k)} \varphi_j^* g(x)\end{aligned}$$

where

$$a_k = \begin{cases} 2^{-([\alpha]+1-\alpha)k} & k \geqslant 0 \\ 2^{(m+\alpha-\lambda)k} & k < 0. \end{cases} \tag{28}$$

Dividing both sides by $(1+2^k|z-x|)^\lambda$ and taking the supremum over $z \in \mathbb{R}^n$ then gives

$$\psi_k^* g(x) \leqslant C \sum_{j\in\mathbb{Z}} a_{k-j}\, 2^{\alpha(j-k)} \varphi_j^* g(x). \tag{29}$$

Now since $m > \lambda - \alpha$ we can apply Lemma B.2 to obtain

$$\begin{aligned} \Big(\sum_{k\in\mathbb{Z}} \big(2^{\alpha k} \|\psi_k^* g\|_p\big)^q \Big)^{1/q} &\leqslant C \Big(\sum_{k\in\mathbb{Z}} \Big\| \sum_{j\in\mathbb{Z}} a_{k-j}\, 2^{\alpha j} \varphi_j^* g \Big\|_p^q \Big)^{1/p} \\ &\leqslant C \Big(\sum_{j\in\mathbb{Z}} \big(2^{\alpha j} \|\varphi_j^* g\|_p\big)^q \Big)^{1/p} \\ &= C \Big(\sum_{j\in\mathbb{Z}} \big(2^{\alpha j} \|\varphi_j^* f\|_p\big)^q \Big)^{1/p} \end{aligned}$$

where the last line follows by using the infinite vanishing moments of φ. The Besov-Lipschitz case now follows from the discrete version of Theorem 1.3.

To prove the Triebel-Lizorkin case we similarly combine (29) with Lemma B.3 to obtain

$$\begin{aligned} \Big\| \Big(\sum_{k\in\mathbb{Z}} \big(2^{k\alpha} |\psi_k^* g|\big)^q \Big)^{1/q} \Big\|_p &\leqslant C \Big\| \Big(\sum_{k\in\mathbb{Z}} \Big(\sum_{j\in\mathbb{Z}} a_{k-j} 2^{\alpha j} \varphi_j^* g \Big)^q \Big)^{1/q} \Big\|_p \\ &\leqslant C \Big\| \Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha} \varphi_j^* g\big)^q \Big)^{1/q} \Big\|_p. \end{aligned}$$

As in the Besov-Lipschitz case the discrete version of Theorem 1.3 then completes the proof. □

We make a few remarks about the above proof. Firstly, note that since

$$|\psi_k * g(x)| \leqslant \psi_k^* g(x)$$

the above theorem gives a stronger inequality than just using $\psi_k * g$. Secondly, as remarked after Theorem 2.5, the condition (i) on ψ is only necessary if $\alpha \geqslant 0$. Similarly, by considering the definition of a_k, (28), we see that the condition (ii) is only required when $\alpha \leqslant \lambda$.

The above theorem was similar to the corresponding inequalities in Theorem 1.3. However, as in Theorem 1.3, in the Besov-Lipschitz case we can make a slight improvement by replacing the usual L^p norm by the H^p norm.

Theorem 2.7. *Let $0 < p, q \leqslant \infty$, $\alpha \in \mathbb{R}$ and choose $\lambda \in \mathbb{R}$, $m \in \mathbb{N}$ such that $\lambda > n/p$ and $m > \lambda - \alpha$. Suppose $(1 + |\cdot|)^\lambda \psi(\cdot) \in L^1$ satisfies:*

(i) The kernel ψ has $[\alpha]$ vanishing moments and $(1 + |\cdot|)^{[\alpha]+\lambda+1}\psi(\cdot) \in L^1$.

(ii) We have $\psi \in C^m$ and for each $|\kappa| = m$,

$$(1 + |\cdot|)^\lambda D^\kappa \psi(\cdot) \in L^1.$$

Then for any $f \in \dot{B}^\alpha_{p,q}$ there exists a polynomial ρ such that

$$\Big(\sum_{j\in\mathbb{Z}} \Big(2^{\alpha j}\|\psi_j * (f - \rho)\|_{H^p}\Big)^q\Big)^{1/q} \leqslant C\|f\|_{\dot{B}^\alpha_{p,q}}.$$

Proof. Take any $\phi \in \mathcal{S}$ such that $\widehat{\phi}(0) \neq 0$. We begin by proving that for any $t > 0$ and $x, y \in \mathbb{R}^n$,

$$|\phi_t * \varphi_j * f(y)| \leqslant C\varphi^*_j f(x)(1 + 2^j|x - y|)^\lambda. \tag{30}$$

Suppose $0 < t < 2^{-j}$. Then for any $x, y, z \in \mathbb{R}^n$ we have

$$(1 + 2^j|x - z|)^\lambda \leqslant (1 + 2^j|x - y|)^\lambda\Big(1 + \frac{|y - z|}{t}\Big)^\lambda$$

and hence

$$\begin{aligned} |\phi_t * \varphi_j * f(y)| &\leqslant \int_{\mathbb{R}^n} |\phi_t(y - z)||\varphi_j * f(z)|dz \\ &\leqslant \varphi^*_j f(x)(1 + 2^j|x - y|)^\lambda \int_{\mathbb{R}^n} |\phi_t(z)|\Big(1 + \frac{|z|}{t}\Big)^\lambda dz \\ &\leqslant C\varphi^*_j f(x)(1 + 2^j|x - y|)^\lambda. \end{aligned}$$

On the other hand suppose $2^{-j} \leqslant t$ and take $\mu \in \mathcal{S}$ such that

$$\widehat{\mu}(\xi) = \begin{cases} 1 & 1/2 \leqslant |\xi| \leqslant 2 \\ 0 & |\xi| \leqslant 1/4. \end{cases}$$

Then firstly $\varphi_j = \varphi_j * \mu_j$ for any $j \in \mathbb{Z}$ and secondly μ has infinite vanishing moments. By Lemma 2.4 the vanishing moments on μ imply, for any $m \in \mathbb{N}$,

$$\begin{aligned} \int_{\mathbb{R}^n} |\mu_j * \phi_t(z)(1 + 2^j|z|)^\lambda dz &\leqslant \Big(\frac{2^{-j}}{t}\Big)^{-\lambda} \int_{\mathbb{R}^n} |\mu_{2^{-j}} * \phi_t(z)|\Big(1 + \frac{|z|}{t}\Big)^\lambda dz \\ &\leqslant C\Big(\frac{2^{-j}}{t}\Big)^{m-\lambda}. \end{aligned}$$

Therefore, taking $m > \lambda$, we obtain

$$
\begin{aligned}
|\phi_t * \varphi_j * f(y)| &= |\phi_t * \mu_j * \varphi_j * f(y)| \\
&\leqslant \int_{\mathbb{R}^n} |\mu_j * \phi_t(y-z)||\varphi_j * f(z)|dz \\
&\leqslant \varphi_j^* f(x)(1+2^j|x-y|)^\lambda \int_{\mathbb{R}^n} |\mu_j * \phi_t(z)|(1+2^j|z|)^\lambda dz \\
&\leqslant C\varphi_j^* f(x)(1+2^j|x-y|)^\lambda \Big(\frac{2^{-j}}{t}\Big)^{m-\lambda}
\end{aligned}
$$

and hence, as $2^{-j} \leqslant t$, (30) follows.

Now suppose $f \in \dot{B}^\alpha_{p,q}$. By Lemma 2.2 there exists a polynomial ρ such that, for any $\mu \in \mathcal{S}$ with $[\alpha]$ vanishing moments,

$$
\begin{aligned}
|\mu_s * \big(\phi_t * (f-\rho)\big)(x)| &\leqslant \|\phi\|_1 \|\mu_s * (f-\rho)\|_\infty \\
&\leqslant C s^{\alpha - n/p}.
\end{aligned} \tag{31}
$$

Let $g = f - \rho$. The estimate (31) implies that we can use Theorem 2.5, with f replaced by $\phi_t * g$, to obtain

$$
\psi_k * \big(\phi_t * g\big)(x) = \sum_{j\in\mathbb{Z}} \varphi_j * \varphi_j * \psi_k * \phi_t * g(x).
$$

Combining this with (30) we have

$$
\begin{aligned}
|\psi_k * \big(\phi_t * g\big)(x)| &\leqslant \sum_{j\in\mathbb{Z}} |\varphi_j * \varphi_j * \psi_k * \phi_t * g(x)| \\
&\leqslant \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} |\varphi_j * \psi_k(x-y)|\, |\phi_t * \varphi_j * g(y)| dy \\
&\leqslant C \sum_{j\in\mathbb{Z}} \varphi_j^* g(x) \int_{\mathbb{R}^n} |\varphi_j * \psi_k(x-y)|(1+2^j|x-y|)^\lambda dy
\end{aligned}
$$

and hence, by taking $z = x$ in Lemma 2.6, we see that

$$
|\phi_t * (\psi_k * g)(x)| \leqslant C \sum_{j\in\mathbb{Z}} a_{k-j} 2^{\alpha(j-k)} \varphi_j^* g(x)
$$

where

$$
a_k = \begin{cases} 2^{-([\alpha]+1-\alpha)k} & k \geqslant 0 \\ 2^{(m+\alpha-\lambda)k} & k < 0. \end{cases}
$$

Thus taking the supremum over $t > 0$ we obtain

$$
\sup_{t>0} |\phi_t * (\psi_k * g)(x)| \leqslant C \sum_{j\in\mathbb{Z}} a_{k-j} 2^{\alpha(j-k)} \varphi_j^* g(x).
$$

The result now follows by using a similar argument to that used to prove Theorem 2.1. $\square$

CHAPTER 3

Sufficient Conditions

We now turn to the problem of finding sufficient conditions on a distribution f to belong to the Besov-Lipschitz or Triebel-Lizorkin spaces using the kernel ψ. In particular we aim to prove inequalities of the form

$$\|f\|_{\dot{B}^{\alpha}_{p,q}} \leqslant C\Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}\|\psi_j * f\|_p\big)^q\Big)^{1/q} \tag{32}$$

and

$$\|f\|_{\dot{F}^{\alpha}_{p,q}} \leqslant C\Big\|\Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}|\psi_j * f|\big)^q\Big)^{1/q}\Big\|_p \tag{33}$$

without the assumption $\psi \in \mathcal{S}$. As we will see this is possible in some special cases but in the general case the difficulty of giving a pointwise definition to the convolution $\psi * f$ will force us to adopt a maximal function version of the above inequalities.

1. Preliminary Results

As in Chapter 2, the Calderón reproducing formula will be an essential tool in the proof of the inequalities (32) and (33). In this section we present a version of the Calderón reproducing formula that, while similar to the previous version proved in Chapter 2, has differing assumptions and conclusions. One such assumption that will be used is the Tauberian condition; see (35).

The Tauberian condition is the simplest and perhaps most general condition required to guarantee that a Calderón type formula exists. Its usefulness comes from the fact that for a given function $\psi \in L^1$ satisfying the Tauberian condition, there exists a function η such that

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi)\widehat{\eta}(2^{-j}\xi) = 1. \tag{34}$$

The Calderón representation formula will then follow from (34) by a similar line of proof as that used to prove Theorem 2.5. Before we come to the proof of these assertions we remark that the proofs and presentation of the Calderón reproducing formula given in this section have been considerably influenced by [**8**], [**9**], and [**14**].

For the reader's convenience we restate the Tauberian condition. We say $\psi \in L^1$ satisfies the *Tauberian condition* if, for every $|\xi| = 1$, there exist $c > 0$ and $0 < 2\sigma \leqslant \varrho < \infty$ such that for each $\sigma < t < \varrho$

$$|\widehat{\psi}(t\xi)|^2 \geqslant c > 0. \tag{35}$$

Note that the Tauberian condition is essentially a non-degeneracy condition, since it allows us to deduce that the family $\big(\widehat{\psi}(2^{-j}\xi)\big)_{j\in\mathbb{Z}}$ does not simultaneously vanish for any $\xi \neq 0$. This observation allows us to prove the following lemma.

Lemma 3.1. *Fix $N \in \mathbb{N}$ with $N > n$. Suppose $\psi \in L^1$ satisfies the Tauberian condition and $\widehat{\psi} \in C^N\big(\mathbb{R}^n \setminus \{0\}\big)$. Then there exists $\widehat{\eta} \in C^N$ such that for any $\xi \neq 0$ we have*

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi)\widehat{\eta}(2^{-j}\xi) = 1.$$

Moreover, $|\eta(x)| \leqslant C(1 + |x|)^{-N}$ for all $x \in \mathbb{R}^n$, and supp $\widehat{\eta} \subseteq \{a \leqslant |\xi| \leqslant b\}$ for some $0 < 2a \leqslant b < \infty$.

Proof. The continuity of $\widehat{\psi}$ together with the Tauberian condition implies that for each $|\xi_0| = 1$ there exists $c_0, r_0 > 0$ and $0 < 2a_0 \leqslant b_0 < \infty$ such that for any $a_0 < t < b_0$ we have

$$|\widehat{\psi}(t\xi)|^2 \geqslant c_0 \qquad \forall\ \xi \in B(\xi_0, r_0) \cap S^{n-1},$$

where $S^{n-1} = \{|\xi| = 1\}$. Hence, using the compactness of the set S^{n-1}, we obtain a finite set $\{\xi_1, ..., \xi_m\}$ of points of S^{n-1} such that $S^{n-1} \subset \cup_{1\leqslant i\leqslant m} B_{r_i}(\xi_i)$. Taking $c = \min_i \{c_i\}$, $a = \min_i \{a_i\}$ and $b = \max_i \{b_i\}$, we have $0 < 2a \leqslant b$ and $c > 0$ satisfying

$$|\widehat{\psi}(t\xi)|^2 \geqslant c \qquad \forall\ \xi \in S^{n-1} \tag{36}$$

for any $a < t < b$.

Take $\phi \in C^\infty(\mathbb{R})$ such that $\phi \geqslant 0$ and

$$\phi(s) = \begin{cases} 1 & s \in [a, b] \\ 0 & s \in \mathbb{R} \setminus [a/2, 2b]. \end{cases}$$

For $|\xi| > 0$ let

$$\rho(\xi) = \sum_{j\in\mathbb{Z}} \phi(2^{-j}|\xi|)|\widehat{\psi}(2^{-j}\xi)|^2.$$

Since ϕ has compact support this is a finite sum and $\rho(2^{-1}\xi) = \rho(\xi)$. Moreover from (36) and the fact that $2a \leqslant b$ we have $\rho(\xi) > 0$ for any $\xi \neq 0$. Define

$$\widehat{\eta}(\xi) = \begin{cases} \phi(|\xi|)\overline{\widehat{\psi}(\xi)}\rho(\xi)^{-1} & \xi \neq 0 \\ 0 & \xi = 0. \end{cases}$$

Then supp $\widehat{\eta} \subseteq \{a/2 \leqslant |\xi| \leqslant 2b\}$ and the condition $\widehat{\psi} \in C^N\big(\mathbb{R}^n \setminus \{0\}\big)$ implies $\widehat{\eta} \in C^N$. Thus by the Fourier inversion theorem (and integration by parts) we have, for every $x \in \mathbb{R}^n$,

$$|\eta(x)| \leqslant C(1+|x|)^{-N}$$

and therefore, as $N > n$, $\eta \in L^1$. Finally, for any $|\xi| \neq 0$,

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi)\widehat{\eta}(2^{-j}\xi) &= \frac{1}{\rho(\xi)} \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^{-j}\xi)|^2 \widehat{\phi}(2^{-j}\xi) \\ &= 1.\end{aligned}$$

□

Using this lemma then gives the following Calderón type formula.

PROPOSITION 3.2. *Fix $\ell \in \mathbb{N}$, $m \in \mathbb{Z}$ and suppose $(1+|\cdot|)^{-\ell} g(\cdot) \in L^\infty \cap C^0$. Assume $\psi \in L^1$ satisfies the Tauberian condition (see (35)) and $\widehat{\psi} \in C^{n+1+\ell}\big(\mathbb{R}^n \setminus \{0\}\big)$. Then there exists $\phi \in L^1$ such that, for every $x \in \mathbb{R}^n$, we have $|\phi(x)| \leqslant C(1+|x|)^{-(n+1+\ell)}$, and*

$$g(x) = \phi_m * g(x) + \sum_{j=m+1}^{\infty} \big(\psi_j * \eta_j\big) * g(x), \tag{37}$$

where η is as in Lemma 3.1. Moreover $\widehat{\phi} \in C^{n+1+\ell}$ and $\widehat{\phi}(0) = 1$ with supp $\widehat{\phi}$ compact.

PROOF. Define, for $\xi \neq 0$,

$$\widehat{\phi}(\xi) = \sum_{j=-\infty}^{0} \widehat{\eta}(2^{-j}\xi)\widehat{\psi}(2^{-j}\xi)$$

and take $\widehat{\phi}(0) = 1$ where η is as in Lemma 3.1. Then since $\widehat{\eta}$ has compact support this is a finite summation and $\widehat{\phi}(\xi) = 0$ whenever $|\xi| > b$. Similarly for $|\xi| < a$ we see that

$$\widehat{\phi}(\xi) = \sum_{j=-\infty}^{0} \widehat{\eta}(2^{-j}\xi)\widehat{\psi}(2^{-j}\xi) = \sum_{j=-\infty}^{\infty} \widehat{\eta}(2^{-j}\xi)\widehat{\psi}(2^{-j}\xi) = 1.$$

Now as $\widehat{\eta}\widehat{\psi} \in C^{n+1+\ell}$ we have $\widehat{\phi} \in C^{n+1+\ell}$ and therefore, for every $x \in \mathbb{R}^n$,

$$|\phi(x)| \leqslant \frac{C}{(1+|x|)^{n+1+\ell}}$$

and

$$|\eta_j * \psi_j(x)| \leqslant \frac{C}{(1+|x|)^{n+1+\ell}}.$$

Moreover, for any $\xi \in \mathbb{R}^n$ and $m, M \in \mathbb{Z}$ with $M > m$,

$$\sum_{j=m+1}^{M} \widehat{\eta}(2^{-j}\xi)\widehat{\psi}(2^{-j}\xi) = \sum_{j=-\infty}^{M} \widehat{\eta}(2^{-j}\xi)\widehat{\psi}(2^{-j}\xi) - \sum_{j=-\infty}^{m} \widehat{\eta}(2^{-j}\xi)\widehat{\psi}(2^{-j}\xi)$$
$$= \widehat{\phi}(2^{-M}\xi) - \widehat{\phi}(2^{-m}\xi).$$

Hence by the Fourier inversion theorem we obtain, for any $x \in \mathbb{R}^n$,

$$\sum_{j=m+1}^{M} \eta_j * \psi_j(x) = \phi_M(x) - \phi_m(x). \tag{38}$$

The growth condition on g and the decay of ϕ and $\eta * \psi$ imply that the convolutions $\phi * g$ and $\big(\psi * \eta\big) * g$ exist. Taking the convolution of both sides of (38) with $g \in C^0$, using the fact that the family $(\phi_M)_{M\in\mathbb{N}}$ forms an approximation to the identity, and letting M tend to infinity then gives, for every $x \in \mathbb{R}^n$,

$$g(x) = \phi_m * g(x) + \sum_{j=m+1}^{\infty} \big(\psi_j * \eta_j\big) * g(x).$$

□

Note that the assumption $(1 + |\cdot|)^{-\ell} g(\cdot) \in L^\infty$ can be replaced with the assumption that $g \in L^1$. Since then again, the convolutions $\phi * g$ and $(\eta * \psi) * g$ exist. Moreover we have $\lim_{M\to\infty} \phi_M * g(x) = g(x)$. Thus the assumption $g \in L^1$ in the above proposition is also sufficient.

We will need the above proposition to estimate the Peetre maximal function $\psi_k^* f$, in terms of the Hardy-Littlewood maximal function $M(\psi_k * f)$; see Theorem 3.9. However, in the special case we will consider in the next section, we can prove a full Calderón type formula.

Lemma 3.3. *Fix $\ell, N \in \mathbb{N}$ and take $f \in \mathcal{S}'$ such that $D^\kappa f$ is a bounded distribution for each $|\kappa| = \ell$. Suppose $\nu \in \mathcal{S}$ has ℓ vanishing moments. Assume $\psi \in L^1$ satisfies the Tauberian condition (see (35)) and $\widehat{\psi} \in C^{n+1+N}\big(\mathbb{R}^n \setminus \{0\}\big)$. Then for every $x \in \mathbb{R}^n$ we have*

$$\nu * f(x) = \sum_{j=-\infty}^{\infty} \psi_j * \eta_j * \big(\nu * f\big)(x) \tag{39}$$

where η is as in Lemma 3.1.

Proof. Firstly, by Lemma 1.8, the ℓ vanishing moments of ν imply $\nu * f \in L^\infty \cap C^\infty$. Therefore by the previous proposition we have, for every $x \in \mathbb{R}^n$ and any $m \in \mathbb{Z}$,

$$\nu * f(x) = \phi_m * (\nu * f)(x) + \sum_{j=m+1}^{\infty} \psi_j * \eta_j * (\nu * f)(x). \tag{40}$$

where η and ϕ are as in Lemmata 3.1 and 3.2 respectively. Note that the compact support of $\widehat{\phi} \in C^{n+1+N}$ implies that $\phi \in C^{\infty} \cap L^1$ and $\partial_{x_i}\phi \in L^1$ for each $i = 1, ..., n$.

Since $\nu \in \mathcal{S}$ has ℓ vanishing moments Theorem 1.1 gives $\nu^{\gamma} \in \mathcal{S}$ such that

$$\nu = \sum_{|\gamma|=\ell+1} D^{\gamma}\nu^{\gamma} \tag{41}$$

Take any $|\gamma| = \ell+1$. Then there exists a multi-index κ with $|\kappa| = \ell$ such that $D^{\gamma} = \partial_{x_i} D^{\kappa}$ for some $i = 1, ..., n$. Now using integration by parts we have

$$\begin{aligned} |\phi_m * (D^{\gamma}\nu^{\gamma} * f)(x)| &= 2^m (\partial_{x_i}\phi)_m * (D^{\kappa}\nu^{\gamma} * f)(x)| \\ &\leqslant 2^m \|\partial_{x_i}\phi\|_1 \|D^{\kappa}\nu^{\gamma} * f\|_{\infty} \end{aligned}$$

and hence letting m tend to $-\infty$ we see that

$$\lim_{m\to-\infty} \|\phi_m * (D^{\gamma}\nu^{\gamma})\|_{\infty} = 0.$$

Therefore, combining (40) and (41) gives, for every $x \in \mathbb{R}^n$,

$$\nu * f(x) = \sum_{j=-\infty}^{\infty} \psi_j * \eta_j * (\nu * f)(x)$$

as required.

□

2. Maximal Function Version

As we have mentioned previously, the first obstacle in the proof of an inequality like

$$\|f\|_{\dot{F}^{\alpha}_{p,q}} \leqslant C \Big\| \Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}|\psi_j * f|\big)^{1/q} \Big\|_p \tag{42}$$

is the definition of the convolution $\psi_j * f$, where f is a distribution and ψ is our general kernel. As in the section on bounded distributions (Chapter 2, Section 4), a partial solution to this problem can be found by restricting ourselves to bounded distributions. We remind the reader that $f \in \mathcal{S}'$ is a *bounded distribution* if, for any $\phi \in \mathcal{S}$,

$$\phi * f \in L^{\infty}.$$

Also recall that if $f \in \mathcal{S}'$ and $D^{\kappa} f$ is a bounded distribution for every $|\kappa| = \ell$, we can define the convolution $\psi * f$ as a distribution by

$$\psi * f(\phi) = \int_{\mathbb{R}^n} \widetilde{\phi} * f(x)\, \widetilde{\psi}(x)\, dx$$

as long as $(1 + |\cdot|)^{\ell}\psi(\cdot) \in L^1$ (where $\ell \in \mathbb{N}$). Note that the integral converges absolutely since the conditions on f imply $|\phi * f(x)| \leqslant C(1 + |x|)^{\ell}$ for any $\phi \in \mathcal{S}$.

Despite the fact that we can define the convolution $\psi_j * f$ as a distribution, we are still left with the problem that we need a pointwise definition of $\psi_j * f$ for an inequality like

$$\|f\|_{\dot{F}^{\alpha}_{p,q}} \leqslant C\Big\|\Big(\sum_{j\in\mathbb{Z}}\big(2^{j\alpha}|\psi_j * f|\big)^{1/q}\Big\|_p$$

to make sense. This problem is a serious one as without additional assumptions $\psi_j * f$ is not necessarily a function. For example, when $n = 1$ take $f = \partial_x\delta_0$ and

$$\psi(x) = \begin{cases} 1 & |x| \leqslant 1 \\ 0 & |x| > 1 \end{cases}$$

where δ_a is the dirac delta distribution at a, i.e. $\delta_a(\phi) = \phi(a)$. Then it is easy to see f is a bounded distribution and $\psi \in L^1$. But for any $\phi \in \mathcal{S}$ we have

$$(\psi * f)(\phi) = \int_0^1 \partial_x\phi(y)dy = \phi(1) - \phi(0)$$

which implies $\psi * f = \delta_1 - \delta_0$. Therefore $\psi * f$ is not a function.

We have yet to find a completely satisfactory condition on ψ to guarantee that $\psi * f$ is a function (though we mention a few possible conditions in the next section). A method of avoiding the problem of a pointwise definition of $\psi * f$ is to replace the distribution $\psi * f$ with an appropriately defined maximal function, which does take on pointwise values. This is the approach we shall take in this section.

We remind the reader that we have fixed a $\varphi \in \mathcal{S}$ satisfying supp $\widehat{\varphi} \subseteq \{1/2 \leqslant |\xi| \leqslant 2\}$ and $\sum_{j\in\mathbb{Z}}\widehat{\varphi}(2^{-j}\xi)^2 = 1$. Take $G \in \mathcal{S}$ to be the Gaussian kernel

$$\widehat{G}(\xi) = e^{-\pi|\xi|^2}$$

and define, for $\lambda > 0, f \in \mathcal{S}'$, and $k \in \mathbb{Z}$,

$$\big(\psi_k * f\big)^{**}_{\lambda}(x) = \sup_{y\in\mathbb{R}^n}\frac{|G_k * (\psi_k * f)(y)|}{(1 + 2^k|x-y|)^{\lambda}}.$$

Then we have the following result.

Lemma 3.4. *Fix $\ell, \lambda \in \mathbb{N}$ and take any $f \in \mathcal{S}'$ such that $D^{\kappa}f$ is a bounded distribution for each $|\kappa| = \ell$. Suppose ψ satisfies the Tauberian condition, $\widehat{\psi} \in C^{n+1+\lambda}\big(\mathbb{R}^n \setminus \{0\}\big)$ and $(1 + |\cdot|)^{\ell}\psi(\cdot) \in L^1$. Then there exists $s > 0$ such that, for every $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$,*

$$|\varphi_j * f(x)| \leqslant C\sum_{|j-k|\leqslant s}\big(\psi_k * f\big)^{**}_{\lambda}(x).$$

PROOF. Using Lemma 3.3, noting that by Lemma 1.6 we can rearrange the convolution, we have for every $x \in \mathbb{R}^n$,

$$\varphi_j * f(x) = \sum_{k=-\infty}^{\infty} \eta_k * \varphi_j * \psi_k * f(x)$$

where η is as in Lemma 3.1. Since the supports of both $\widehat{\eta}$ and $\widehat{\varphi}$ are contained in some annulus about the origin, we can find $s > 0$ such that $\eta_k * \varphi_j = 0$ for any $|j - k| > s$ and therefore

$$\varphi_j * f(x) = \sum_{|j-k| \leqslant s} \eta_k * \varphi_j * \psi_k * f(x). \tag{43}$$

Take $\rho \in \mathcal{S}$ such that

$$\rho(x) = \begin{cases} 1 & |x| \leqslant 2^{s+1} \\ 0 & |x| \geqslant 2^{s+2} \end{cases} \tag{44}$$

and define $\mu \in \mathcal{S}$ by

$$\widehat{\mu}(\xi) = \frac{\rho(\xi)}{\widehat{G}(\xi)} = \rho(\xi)\, e^{\pi|\xi|^2}.$$

Suppose $|j - k| \leqslant s$ and

$$\xi \in \operatorname{supp}\ \widehat{\varphi_j} \subseteq \{1/2 \leqslant 2^{-j}|\xi| \leqslant 2\}.$$

Then since $2^{-k}\,|\xi| \leqslant 2^s\, 2^{-j}\,|\xi| \leqslant 2^{s+1}$ we see that $\rho(2^{-k}\xi) = 1$ on the support of $\widehat{\varphi_j}$. Hence for any $|j - k| \leqslant s$ we have

$$\widehat{\varphi}(2^{-j}\xi) = \widehat{\varphi}(2^{-j}\xi)\, \rho(2^{-k}\xi) = \widehat{\varphi}(2^{-j}\xi)\, \widehat{\mu}(2^{-k}\xi)\, \widehat{G}(2^{-k}\xi).$$

Therefore, as $|\eta(x)| \leqslant C(1 + |x|)^{-(n+1+\lambda)}$ and $\mu, \varphi \in \mathcal{S}$, we have for any $x \in \mathbb{R}^n$

$$\begin{aligned} |\varphi_j * f(x)| &\leqslant \sum_{|j-k| \leqslant s} \big|\eta_k * \big(\varphi_j * \psi_k * f\big)(x)\big| \\ &\leqslant \sum_{|j-k| \leqslant s} \big|\eta_k * \mu_k * \varphi_j * \big(G_k * \psi_k * f\big)(x)\big| \\ &\leqslant \sum_{|j-k| \leqslant s} \big(\psi_k * f\big)^{**}_{\lambda}(x) \int_{\mathbb{R}^n} |\eta_k * \mu_k * \varphi_j(y)|\, (1 + 2^k|y|)^{\lambda} dy \\ &\leqslant C \sum_{|j-k| \leqslant s} \big(\psi_k * f\big)^{**}_{\lambda}(x). \end{aligned}$$

□

The use of the Gaussian function to define $(\psi_k * f)^{**}_\lambda$ was not essential. In fact, a close examination of the above proof shows we need only take a $\phi \in \mathcal{S}$ with $|\widehat{\phi}(\xi)| > c > 0$ for any $a/4 \leqslant |\xi| \leqslant 4b$, where a and b are as in Lemma 3.1. For such a ϕ, the above result would still hold with G replaced by ϕ. This follows by noting that for η, a and b as in Lemma 3.1 we have

$$\eta_k * \varphi_j = 0$$

whenever[1] $2^{j-k} \leqslant a/2$ or $2^{j-k} \geqslant 2b$. Next, instead of using (44) to define ρ, we take $\rho \in \mathcal{S}$ with $\rho = 1$ on $a/4 \leqslant |\xi| \leqslant 4b$ and support sufficiently close to $\{a/4 \leqslant |\xi| \leqslant 4b\}$. Then as above we have $\rho(2^{-k}\xi) = 1$ on the support of $\widehat{\varphi}_j\widehat{\eta}_k$ and moreover $\mu \in \mathcal{S}$, where μ is defined as in the above lemma. The remainder of the proof would then follow as above. However, as ϕ would then depend on the support of $\widehat{\eta}$, it is more convenient for the statement of the above lemma (and of the following theorem) to use the Gaussian kernel instead.

Finally we come to the following theorem giving sufficient conditions for inclusion in the spaces $\dot{B}^\alpha_{p,q}$ and $\dot{F}^\alpha_{p,q}$ using the maximal function $(\psi_k * f)^{**}_\lambda$. Its proof is a straight forward application of Lemma 3.4 combined with the Lemma B.2 (for the Besov-Lipschitz case) and Lemma B.3 (for the Triebel-Lizorkin case).

Theorem 3.5. *Fix $\ell, \lambda \in \mathbb{N}$. Suppose $D^\kappa f$ is a bounded distribution for any $|\kappa| = \ell$ and assume ψ satisfies the Tauberian condition, $\widehat{\psi} \in C^{n+1+\lambda}\big(\mathbb{R}^n \setminus \{0\}\big)$ and $(1+|\cdot|)^\ell \psi(\cdot) \in L^1$. Then for any $0 < p, q \leqslant \infty$, $\alpha \in \mathbb{R}$ and $\lambda > 0$ we have*

$$\|f\|_{\dot{B}^\alpha_{p,q}} \leqslant C\Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}\|\big(\psi_j * f\big)^{**}_\lambda\|_p\big)^q\Big)^{1/q}$$

and if $p < \infty$

$$\|f\|_{\dot{F}^\alpha_{p,q}} \leqslant C\Big\|\Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}\big(\psi_j * f\big)^{**}_\lambda\big)^q\Big)^{1/q}\Big\|_p.$$

Proof. Lemma 3.4 together with Lemma B.2 gives

$$\|f\|_{\dot{B}^\alpha_{p,q}} = \Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}\|\varphi_j * f\|_p\big)^q\Big)^{1/q} \leqslant \Big(\sum_{j\in\mathbb{Z}} \Big\| C \sum_{|j-k|<s} 2^{(j-k)\alpha}\, 2^{k\alpha}\big(\psi_k * f\big)^{**}_\lambda\Big\|_p^q\Big)^{1/q}$$
$$\leqslant C\Big(\sum_{k\in\mathbb{Z}} \big(2^{k\alpha}\|\big(\psi_k * f\big)^{**}_\lambda\|_p\big)^q\Big)^{1/q}.$$

[1]This observation also gives an explicit value for s, namely $s = \frac{\max\{\,|\ln a/2|, |\ln 2b|\}}{\ln 2}$.

Similarly for the Triebel-Lizorkin case we again use Lemma 3.4 and apply Lemma B.3 to obtain

$$\|f\|_{\dot F^{\alpha}_{p,q}} = \Big\|\Big(\sum_{j\in\mathbb{Z}}\big(2^{j\alpha}\varphi_j * f\big)^q\Big)^{1/q}\Big\|_p \leqslant \Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{|j-k|\leqslant s} 2^{(j-k)\alpha}2^{k\alpha}\big(\psi_k * f\big)^{**}_{\lambda}\Big)^q\Big)^{1/q}\Big\|_p$$
$$\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}\big(2^{j\alpha}\big(\psi_k * f\big)^{**}_{\lambda}\big)^q\Big)^{1/q}\Big\|_p.$$

□

Consider the example we gave at the beginning of this section, namely $n=1$, $f=\partial_x\delta_0$ and

$$\psi(x) = \begin{cases} 1 & |x|>1 \\ 0 & |x|\leqslant 1 \end{cases}$$

where δ_a is the dirac delta distribution at a, i.e. $\delta_a(\phi)=\phi(a)$. Then ψ and f satisfy the assumptions of the previous theorem but $\psi * f$ is not a function. Thus a maximal type formulation, or some other method of evaluating the l^q and L^p norms of a distribution, is essential. In the sense that without additional assumptions, we cannot hope for an inequality of the form

$$\|f\|_{\dot F^{\alpha}_{p,q}} \leqslant \Big\|\Big(\sum_{j\in\mathbb{Z}}\big(2^{j\alpha}|\psi_j * f|\big)^q\Big)^{1/q}\Big\|_p$$

since such an inequality has no meaning if $\psi_j * f$ is not a function.

To conclude this section we note that by the H^p characterisation of Feffermen and Stein, [**12**, pg 92], we have

$$\|(\psi_k * f)^{**}_{\lambda}\|_p = \big\|\sup_{y\in\mathbb{R}^n}|G_k * \psi_k * f(\cdot - y)|(1+2^k|y|)^{-\lambda}\big\|_p$$
$$\leqslant \big\|\sup_{y\in\mathbb{R}^n, t>0}|G_t * \psi_k * f(\cdot - y)|(1+|y/t|)^{-\lambda}\big\|_p$$
$$\leqslant C\|\psi_k * f\|_{H^p}$$

for any $\lambda > n/p$. Thus in the Besov-Lipschitz case we can restate the previous theorem in the following form.

THEOREM 3.6. *Fix $0<p,q\leqslant\infty$, $\alpha\in\mathbb{R}$ and $\ell,\lambda\in\mathbb{N}$. Suppose $D^{\kappa}f$ is a bounded distribution for any $|\kappa|=\ell$ and assume ψ satisfies the Tauberian condition, $\widehat{\psi}\in C^{n+1+\lambda}\big(\mathbb{R}^n\setminus\{0\}\big)$, and $(1+|\cdot|)^{\ell}\psi(\cdot)\in L^1$. Then if $\lambda>n/p$ we have*

$$\|f\|_{\dot B^{\alpha}_{p,q}} \leqslant C\Big(\sum_{j\in\mathbb{Z}}\big(2^{j\alpha}\|\psi_j * f\|_{H^p}\big)^q\Big)^{1/q}.$$

3. Special Cases

As remarked previously, the use of the maximal function $(\psi_k * f)^{**}_\lambda$ in the previous section was essential, due to the fact that with the assumptions made we could not define the convolution $\psi_k * f$ as a function. However, in many special cases we can actually assign a pointwise value to $\psi_k * f$. In this case it would be desirable to obtain a version of Theorem 3.5 with the maximal function $(\psi_k * f)^{**}_\lambda$ replaced with the function $\psi_k * f$. To make the following results as general as possible, we will make the following assumptions.

(i) We suppose, as usual, that $f \in \mathcal{S}'$ and $D^\kappa f$ is a bounded distribution for any $|\kappa| = \ell$.

(ii) The kernel ψ should satisfy $(1 + |\cdot|)^\ell \psi(\cdot) \in L^1$.

(iii) For each $j \in \mathbb{Z}$ the distribution $\psi_j * f$ is a function such that, for any $x \in \mathbb{R}^n$,

$$|\psi_j * f(x)| \leqslant C(1 + |x|)^\ell.$$

We remark that in (iii) the constant C may depend on j but not x.

Assumptions (i) and (ii) guarantee that $\psi_k * f$ can be defined as a distribution and the results of Chapter 1, Section 4 apply. The third assumption is made to simplify matters, as we have yet to come up with a completely satisfactory condition that will imply $\psi_k * f$ is a function. However we can exhibit a number of examples that do satisfy (iii).

For instance, suppose ψ and f satisfy (i) and (ii) and $\widehat{\psi}$ has compact support. Then we can define the convolution $\psi * f$ as a function by letting

$$\psi * f(x) = \psi * (\phi * f)(x)$$

where $\phi \in \mathcal{S}$ satisfies $\widehat{\phi} = 1$ on the support of $\widehat{\psi}$ (so $\psi = \phi * \psi$). It is easy enough to prove that $\psi * f$ is independent of ϕ and moreover agrees with the definition of $\psi * f$ as a distribution.

Another special case we can consider is if $\widehat{\psi} \in C^\infty\big(\mathbb{R}^n \backslash \{0\}\big)$ and $\widehat{\psi}$ is rapidly decreasing. Following E. Stein, [**12**, pg 90], let $\rho \in \mathcal{S}$ satisfy $\widehat{\rho}(\xi) = 1$ on some ball about the origin. Then

$$\widehat{\psi} = \widehat{\rho}\widehat{\psi} + (1 - \widehat{\rho})\widehat{\psi}$$

and thus

$$\psi(x) = \rho * \psi(x) + \phi(x)$$

where $\widehat{\phi} = (1 - \widehat{\rho})\widehat{\psi} \in \mathcal{S}$. Therefore, provided of course that ψ and $f \in \mathcal{S}'$ satisfy (i) and (ii), we can define

$$\psi * f(x) = \psi * (\rho * f)(x) + \phi * f(x).$$

From this definition is easy to see that $\psi * f$ is independent of ρ, agrees with the definition of $\psi * f$ as a distribution, and that the inequality (iii) holds. This shows that in particular, if K is the Poisson kernel,

$$\widehat{K}(\xi) = e^{-2\pi|\xi|},$$

and $\widehat{\psi}(\xi) = |\xi|^{2k}\widehat{K}(\xi)$ with $k \in \mathbb{N}$ and $2k > \ell$, then ψ satisfies[2] (ii) and moreover $\widehat{\psi} \in C^\infty\big(\mathbb{R}^n \setminus \{0\}\big)$ is rapidly decreasing. Thus for any $f \in \mathcal{S}'$ satisfying (i) we see that the function $\psi_j * f$ satisfies (iii).

Finally, suppose $f \in \mathcal{S}'$ satisfies the estimate

$$\|\rho_t * f\|_\infty \leqslant Ct^{\alpha - n/p}$$

for any $\rho \in \mathcal{S}$ with $[\alpha]$ vanishing moments. Then, by considering Theorem 2.5, if ψ has $[\alpha]$ vanishing moments and satisfies some additional hypotheses (see Theorem 2.5 for the precise conditions needed) again we see that $\psi * f$ is a function and moreover (iii) holds. Note that this is essentially the same as requiring that $f \in \dot{B}^{\alpha-n/p}_{\infty,\infty}$, at least when working modulo polynomials.

We begin by proving a maximal function free version of Theorem 3.5 in the case $p > 1$ (we may take $p \geqslant 1$ in the Besov-Lipschitz case).

THEOREM 3.7. *Fix $\alpha \in \mathbb{R}$ and $\ell \in \mathbb{N}$. Assume ψ satisfies the Tauberian condition, $\widehat{\psi} \in C^{n+1}\big(\mathbb{R}^n \setminus \{0\}\big)$ and $(1 + |\cdot|)^\ell \psi(\cdot) \in L^1$. Take $f \in \mathcal{S}'$ such that $D^\kappa f$ is a bounded distribution for every $|\kappa| = \ell$ and suppose that for every $j \in \mathbb{Z}$ the distribution $\psi_j * f$ is a locally integrable function.*

Then for any $1 \leqslant p \leqslant \infty$, $0 < q \leqslant \infty$ there exists a constant C, independent of f, such that

$$\|f\|_{\dot{B}^\alpha_{p,q}} \leqslant C\Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}\|\psi_j * f\|_p\big)^q\Big)^{1/q} \tag{45}$$

and if $1 < p < \infty$, $1 < q \leqslant \infty$,

$$\|f\|_{\dot{F}^\alpha_{p,q}} \leqslant C\Big\|\Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}|\psi_j * f|\big)^q\Big)^{1/q}\Big\|_p. \tag{46}$$

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[2]This follows from the equality

$$\psi(x) = \int_{\mathbb{R}^n} |\xi|^{2k} e^{-2\pi|x|} e^{2\pi i x\cdot\xi}\, d\xi = (\Delta)^k \frac{C}{(1+|x|^2)^{(n+1)/2}},$$

where $\Delta = \sum_{j=1\ldots n} \partial^2_{x_j}$ is the Laplacian on $\mathbb{R}^n$.

PROOF. An application of Lemma 3.3 (with $N = 0$) gives, for every $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$,

$$\varphi_j * f(x) = \sum_{|j-k|\leqslant s} \eta_k * \psi_k * \big(\varphi_j * f\big)(x).$$

Following the argument from Lemma 3.4 we have an $s > 0$ such that

$$\varphi_j * f(x) = \sum_{|j-k|\leqslant s} \eta_k * \psi_k * \big(\varphi_j * f\big)(x). \tag{47}$$

Thus using the decay of η and φ we have for every $x \in \mathbb{R}^n$ the pointwise inequality

$$\begin{aligned}
|\varphi_j * f(x)| &\leqslant \sum_{|j-k|\leqslant s} |\eta_k * \varphi_j * \psi_k * f(x)| \\
&\leqslant C \sum_{|j-k|\leqslant s} \int_{\mathbb{R}^n} \frac{|\psi_k * f(y)|}{(1 + 2^k|x-y)^{n+1}} 2^{kn} dy \\
&\leqslant C \sum_{|j-k|\leqslant s} M(|\psi_k * f|)(x)
\end{aligned}$$

where the last inequality follows by Proposition 1.5. Therefore, assuming $1 < p < \infty$ and $1 < q \leqslant \infty$, we see that by the vector-valued maximal inequality, Theorem 1.4, we have

$$\begin{aligned}
\|f\|_{\dot{F}^{\alpha}_{p,q}} &= \Big\| \Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}|\varphi_j * f|\big)^q \Big)^{1/q} \Big\|_p \\
&\leqslant C \Big\| \Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha} M(|\psi_j * f|)\big)^q \Big)^{1/q} \Big\|_p \\
&\leqslant C \Big\| \Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}|\psi_j * f|\big)^q \Big)^{1/q} \Big\|_p
\end{aligned}$$

and hence the Triebel-Lizorkin case follows.

To prove the Besov-Lipschitz case we see that, assuming $1 \leqslant p \leqslant \infty$, by taking L^p norms of both sides of (47) we obtain via Young's Inequality

$$\|\varphi_j * f\|_p \leqslant \sum_{|j-k|\leqslant s} \|\eta_k * \varphi_j\|_1 \|\psi_k * f\|_p.$$

Hence for any $0 < q \leqslant \infty$ we have

$$\begin{aligned}
\|f\|_{\dot{B}^{\alpha}_{p,q}} &= \Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}\|\varphi_j * f\|_p\big)^q \Big)^{1/q} \\
&\leqslant C \Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}\|\psi_j * f\|_p\big)^q \Big)^{1/q}.
\end{aligned}$$

□

The above theorem gives very general conditions on the kernel ψ. However, note that it still requires that we have some way of defining the distribution $\psi * f$ as a function. Thus, despite the above theorem, we may require ψ to satisfy stronger assumptions. Alternatively of course, we could make further assumptions on the distribution f. However, as our eventual aim is to characterise the spaces $\dot{B}^{\alpha}_{p,q}$ and $\dot{F}^{\alpha}_{p,q}$, we will continue to require as few conditions on f as possible.

We now consider the Triebel-Lizorkin space $\dot{F}^{\alpha}_{1,q}$ and turn to the problem of extending Theorem 3.7 to the case $p = 1$. This extension will again rely heavily on the vector-valued maximal inequality of Fefferman and Stein, Theorem 1.4. However, since Theorem 1.4 fails for $p = 1$, we cannot directly apply it as we did in the proof of Theorem 3.7. Instead we will first need to prove an inequality of the form

$$\|f\|_{\dot{F}^{\alpha}_{1,q}} \leqslant C\Big\|\Big(\sum_{j\in\mathbb{Z}}\big(2^{j\alpha r}M(|\psi_j * f|^r)\big)^{q/r}\Big)^{1/q}\Big\|_1 \tag{48}$$

where $0 < r < \min\{q, 1\}$ and, as previously, $M(g)(x)$ is the Hardy-Littlewood maximal function

$$M(g)(x) = \sup_{r>0}\frac{c_n}{r^n}\int_{|x-y|\leqslant r}|g(y)|dy.$$

The extra factor r is chosen so that q/r and $1/r$ are both greater than 1, this allows us to apply the vector-valued maximal inequality to (48) with p and q in Theorem 1.4 replaced with q/r and $1/r$ respectively. The case $p = 1$ will then follow (see the proof of Theorem 3.10 and Corollary 3.11).

The major difficulty in the extension of Theorem 3.7 to the case $p = 1$ will be the proof of the inequality (48). This will achieved by proving the following technical results. The first is similar to Lemma 2.4 proved earlier while the second was motivated by a discrete version of Theorem 2a from [**14**, pg 61].

LEMMA 3.8. *Fix $N, m > 0$ and suppose η has $N - 1$ vanishing moments and satisfies $|\eta(x)| \leqslant C(1 + |x|)^{-(m+N+n+1)}$. Assume $\psi \in C^m$ satisfies, for each $|\kappa| = m$,*

$$|D^{\kappa}\psi(x)| \leqslant \frac{C}{(1 + |x|)^N}.$$

Then, for any $0 < s \leqslant t$ and $x \in \mathbb{R}^n$,

$$|\eta_s * \psi_t(x)| \leqslant C\Big(\frac{s}{t}\Big)^m t^{-n}\Big(1 + \frac{|x|}{t}\Big)^{-N}.$$

Consequently,

$$\|\eta_s * \psi_t\|_{\infty} \leqslant C\Big(\frac{s}{t}\Big)^m t^{-n}.$$

PROOF. We begin as usual by combining Taylor's formula with the moment condition on η and the differentiability of ψ to obtain, for each $x \in \mathbb{R}^n$,

$$\begin{aligned}
|\eta_s * \psi_t(x)| &= \Big| \int_{\mathbb{R}^n} \eta_s(ty)\, \psi\Big(\frac{x}{t} - y\Big)\, dy \Big| \\
&\leqslant \sum_{|\kappa|=m} c_\kappa \int_{\mathbb{R}^n} |y|^m |\eta_s(ty)| \int_0^1 \rho^{m-1} \Big| D^\kappa \psi\Big(\frac{x}{t} - \rho y\Big)\Big| d\rho\, dy \\
&= \sum_{|\kappa|=m} c_\kappa I_\kappa(x).
\end{aligned}$$

Since $0 < s \leqslant t$ and $\rho \leqslant 1$ we have

$$\begin{aligned}
\Big(1 + \frac{|x|}{t}\Big)^N &\leqslant \Big(1 + \Big|\frac{x}{t} - \rho y\Big|\Big)^N \Big(1 + \rho|y|\Big)^N \\
&\leqslant \Big(1 + \Big|\frac{x}{t} - \rho y\Big|\Big)^N \Big(1 + \frac{t}{s}|y|\Big)^N
\end{aligned}$$

and therefore

$$\begin{aligned}
&\Big(1 + \frac{|x|}{t}\Big)^N I_\kappa(x) \\
&\quad \leqslant \int_{\mathbb{R}^n} |y|^m \Big(1 + \frac{t}{s}|y|\Big)^N |\eta_s(ty)| \int_0^1 \rho^{m-1} \big|D^\kappa \psi(x/t - \rho y)\big| \Big(1 + \Big|\frac{x}{t} - \rho y\Big|\Big)^N d\rho\, dy \\
&\quad \leqslant C\Big(\frac{s}{t}\Big)^m t^{-n} \int_{\mathbb{R}^n} \Big|\frac{t}{s} y\Big|^m \Big(1 + \frac{t}{s}|y|\Big)^N |\eta_{s/t}(y)|\, dy \\
&\quad \leqslant C\Big(\frac{s}{t}\Big)^m t^{-n}
\end{aligned}$$

and hence result follows. $\square$

The following result will prove crucial in the proof of the inequality (48). We remind the reader that the Peetre maximal function, $\psi_j^* f$, is defined by

$$\psi_j^* f(x) = \sup_{z \in \mathbb{R}^n} \frac{|\psi_j * f(z)|}{(1 + 2^j |x - z|)^\lambda}.$$

We will also require the following variation of this maximal function

$$M_{\lambda,m}(x, j) = \sup_{y \in \mathbb{R}^n, k \geqslant j} \frac{|\psi_k * f(y)|}{(1 + 2^j |x - y|)^\lambda}\, 2^{(j-k)m}. \tag{49}$$

Note that if $M_{\lambda,m}(x_0, j)$ is finite for some $x_0 \in \mathbb{R}^n$, then we have $M_{\lambda,m}(x, j) < \infty$ for all $x \in \mathbb{R}^n$. With these definitions at hand we now prove the following theorem.

THEOREM 3.9. *Fix $0 < r \leqslant 1$ and $m, \lambda, \ell \in \mathbb{N}$. Suppose $f \in \mathcal{S}'$ and $D^\kappa f$ is a bounded distribution for each $|\kappa| = \ell$. Assume ψ satisfies the Tauberian condition and furthermore that,* *(i)* $\widehat{\psi} \in C^{n+1+m+\lambda}(\mathbb{R}^n \setminus \{0\})$,

(ii) $(1+|\cdot|)^\ell \psi(\cdot) \in L^1$,

(iii) $\psi \in C^m$ *with*

$$|D^\kappa \psi(x)| \leqslant \frac{C}{(1+|x|)^\lambda}$$

for each $|\kappa| = m$ *and any* $x \in \mathbb{R}^n$.

Let $j \in \mathbb{Z}$ *and suppose that, for every* $k \geqslant j$, *the distribution* $\psi_k * f$ *is a continuous function, and moreover* $M_{\lambda,m}(x_0, j) < \infty$ *for some* $x_0 \in \mathbb{R}^n$. *Then, for any* $x \in \mathbb{R}^n$,

$$\left(\psi_j^* f(x)\right)^r \leqslant C \sum_{k=j}^{\infty} 2^{(j-k)(mr-n)} \int_{\mathbb{R}^n} \frac{|\psi_k * f(y)|^r}{(1+2^j|x-y|)^{\lambda r}} 2^{jn} dy. \tag{50}$$

PROOF. Let ψ and f satisfy the above conditions. The finiteness of $M_{\lambda,m}(x_0, j)$ implies that, for every $u \geqslant j$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} |\psi_u * f(x)| &\leqslant M_{\lambda,m}(x_0, j)(1+2^j|x_0 - x|)^\lambda \, 2^{(u-j)m} \\ &\leqslant C_{x_0,j,u} \, (1+|x|)^\lambda. \end{aligned}$$

Thus we can apply the Calderón formula of Proposition 3.2 to obtain, for every $x \in \mathbb{R}^n$,

$$\psi_u * f(x) = \phi_u * \big(\psi_u * f\big)(x) + \sum_{k=u+1}^{\infty} \big(\eta_k * \psi_k\big) * \big(\psi_u * f\big)(x) \tag{51}$$

where ϕ and η are bounded above by a constant multiple of $(1+|\cdot|)^{-(n+1+\lambda+m)}$. Note that by using Lemma 1.7 we have for any $\mu \in \mathcal{S}$

$$\psi_k * \big(\psi_u * f\big)(\mu) = \psi_u * \big(\psi_k * f\big)(\mu)$$

and hence $\big(\eta_k * \psi_k\big) * \big(\psi_u * f\big)(x) = \big(\eta_k * \psi_u\big) * \big(\psi_k * f\big)(x)$. An application of Lemma 3.8 shows that, for every $x \in \mathbb{R}^n$ and any $u < k$,

$$|\eta_k * \psi_u(x)| \leqslant C 2^{(u-k)m} \frac{2^{un}}{(1+2^u|x|)^\lambda}$$

and thus, for any $u < k$ and every $z \in \mathbb{R}^n$,

$$\begin{aligned} |\eta_k * \psi_k * \big(\psi_u * f\big)(z)| &= |\eta_k * \psi_u * \big(\psi_k * f\big)(z)| \\ &\leqslant C 2^{(u-k)m} \int_{\mathbb{R}^n} \frac{|\psi_k * f(y)|}{(1+2^u|z-y|)^\lambda} 2^{un} dy. \end{aligned}$$

On the other hand, the decay on ϕ shows

$$|\phi_u * \psi_u * f(z)| \leqslant C \int_{\mathbb{R}^n} \frac{|\psi_u * f(y)|}{(1+2^u|z-y|)^\lambda} 2^{un} dy.$$

Therefore, combining these estimates with (51) we obtain, for every $z \in \mathbb{R}^n$ and any $u \geqslant j$,

$$|\psi_u * f(z)| \leqslant C 2^{(u-j)m} \sum_{k=u}^{\infty} 2^{(j-k)m} \int_{\mathbb{R}^n} \frac{|\psi_k * f(y)|}{(1+2^u|z-y|)^\lambda} 2^{un} dy.$$

Now, since $k \geqslant u \geqslant j$, we have

$$\begin{aligned}
&\frac{|\psi_k * f(y)|}{(1+2^u|z-y|)^\lambda}\, 2^{(j-k)m}\, 2^{un} \\
&\quad \leqslant \left(\frac{|\psi_k * f(y)|}{(1+2^j|x-y|)^\lambda}\, 2^{(j-k)m} \right)^r \left(\frac{|\psi_k * f(y)|}{(1+2^j|x-y|)^\lambda}\, 2^{(j-k)m} \right)^{1-r} \frac{(1+2^j|x-y|)^\lambda}{(1+2^u|z-y|)^\lambda}\, 2^{kn} \\
&\quad \leqslant \left(\frac{|\psi_k * f(y)|}{(1+2^j|x-y|)^\lambda}\, 2^{(j-k)m} \right)^r M_{\lambda,m}(x,j)^{1-r}\, (1+2^j|x-z|)^\lambda\, 2^{kn}
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{|\psi_u * f(z)|}{(1+2^j|x-z|)^\lambda}\, 2^{(j-u)m} &\leqslant C M_{\lambda,m}(x,j)^{1-r} \sum_{k=u}^{\infty} 2^{(j-k)mr} \int_{\mathbb{R}^n} \frac{|\psi_k * f(y)|^r}{(1+2^j|x-y|)^{\lambda r}} 2^{kn} dy \\
&\leqslant C M_{\lambda,m}(x,j)^{1-r} \sum_{k=j}^{\infty} 2^{(j-k)(mr-n)} \int_{\mathbb{R}^n} \frac{|\psi_k * f(y)|^r}{(1+2^j|x-y|)^{\lambda r}} 2^{jn} dy.
\end{aligned}$$

Thus taking the supremum over $z \in \mathbb{R}^n$ and $u \geqslant j$ yields

$$M_{\lambda,m}(x,j) \leqslant C M_{\lambda,m}(x,j)^{1-r} \sum_{k=j}^{\infty} 2^{(j-k)(mr-n)} \int_{\mathbb{R}^n} \frac{|\psi_k * f(y)|^r}{(1+2^j|x-y|)^{\lambda r}} 2^{jn} dy.$$

Therefore, provided $M_{\lambda,m}(x,j)$ is finite, we have

$$M_{\lambda,m}(x,j)^r \leqslant C \sum_{k=j}^{\infty} 2^{(j-k)(mr-n)} \int_{\mathbb{R}^n} \frac{|\psi_k * f(y)|^r}{(1+2^j|x-y|)^{\lambda r}} 2^{jn} dy$$

from which the required inequality follows.

□

We are now in a position where we can extend Theorem 3.7 to the case $p = 1$ as most of the hard work in proving the inequality (48) was contained in the previous theorem. However, before we prove the case $p = 1$ we will consider the more general situation $0 < p < 1$ with the assumption $M_{\lambda,m}(x,j) < \infty$.

Theorem 3.10. *Let $0 < p, q \leqslant \infty$, $\alpha \in \mathbb{R}$, and take $\lambda, m, \ell \in \mathbb{N}$ such that $m > \lambda - \alpha$. Suppose $\psi \in L^1$ satisfies;*

(i) The Tauberian condition;

(ii) The differentiability condition $\psi \in C^m$ and $\widehat{\psi} \in C^{n+1+\lambda+m}\big(\mathbb{R}^n \setminus \{0\}\big)$;

(iii) The decay condition $(1+|\cdot|)^\ell\psi(\cdot)\in L^1$ *and*

$$|D^\kappa\psi(x)|\leqslant\frac{C}{(1+|x|)^\lambda}$$

for every $x\in\mathbb{R}^n$ *and any* $|\kappa|=m$.

Assume $f\in\mathcal{S}'$ *and* $D^\kappa f$ *is a bounded distribution for each* $|\kappa|=\ell$. *Moreover, suppose that for each* $j\in\mathbb{Z}$, *the distribution* ψ_j*f *is a continuous function with* $M_{\lambda,m}(x,j)<\infty$. *Then, if* $\lambda>n/p$, *there exists a constant* C *independent of* f *such that*

$$\|f\|_{\dot{B}^\alpha_{p,q}}\leqslant C\Big(\sum_{j\in\mathbb{Z}}\big(2^{j\alpha}\|\psi_j*f\|_p\big)^q\Big)^{1/q}. \tag{52}$$

Similarly, if $\lambda>\max\{n/p,n/q\}$, *then*

$$\|f\|_{\dot{F}^\alpha_{p,q}}\leqslant C\Big\|\Big(\sum_{j\in\mathbb{Z}}(2^{j\alpha}|\psi_j*f|)^q\Big)^{1/q}\Big\|_p. \tag{53}$$

Proof. Following the proof of Theorem 3.7, there exists an $s>0$ such that, for every $x\in\mathbb{R}^n$,

$$\begin{aligned}|\varphi_j*f(x)|&\leqslant\sum_{|j-k|\leqslant s}|\psi_k*\eta_k*\varphi_j*f(x)|\\&\leqslant\sum_{|j-k|\leqslant s}\psi_k^*f(x)\int_{\mathbb{R}^n}\frac{|\eta_k*\varphi_j(y)|}{(1+2^k|y|)^\lambda}\,dy\\&\leqslant C\sum_{|j-k|\leqslant s}\psi_k^*f(x)\end{aligned}$$

where we have used the decay of η to obtain the last inequality. Therefore, summing over $j\in\mathbb{Z}$ yields

$$\sum_{j\in\mathbb{Z}}(2^{j\alpha}|\varphi_j*f(x)|)^q\leqslant C\sum_{j\in\mathbb{Z}}\big(2^{j\alpha}\psi_j^*f(x)\big)^q$$

and hence

$$\|f\|_{\dot{F}^\alpha_{p,q}}\leqslant C\Big\|\Big(\sum_{j\in\mathbb{Z}}(2^{j\alpha}|\psi_j^*f|)^q\Big)^{1/q}\Big\|_p. \tag{54}$$

A similar argument gives

$$\|f\|_{\dot{B}^\alpha_{p,q}}\leqslant C\Big(\sum_{j\in\mathbb{Z}}\big(2^{j\alpha}\|\psi_j^*f\|_p\big)^q\Big)^{1/q}. \tag{55}$$

If $\min\{p,q\}>1$ then Theorem 3.7 gives (52) and (56). Hence we may assume that $\min\{p,q\}\leqslant 1$. Choose $0<r<\min\{p,q\}\leqslant 1$ such that $\lambda>n/r$. The assumption

$M_{\lambda,m}(x,j) < \infty$ allows us to apply Theorem 3.9 with this r to obtain

$$|\psi_j^* f(x)| \leqslant C\Big(\sum_{k=j}^{\infty} 2^{(j-k)(mr-n)} \int_{\mathbb{R}^n} \frac{|\psi_k * f(y)|^r}{(1+2^j|x-y|)^{\lambda r}} 2^{jn} dy\Big)^{1/r}$$
$$\leqslant C\Big(\sum_{k=j}^{\infty} 2^{(j-k)(mr-n)} M(|\psi_k * f|^r)(x)\Big)^{1/r}$$

where the last inequality follows from Proposition 1.5. Combining this with (54) we see that

$$\|f\|_{\dot{F}^{\alpha}_{p,q}} \leqslant \Big\|\Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha}|\psi_k^* f|)^q\Big)^{1/q}\Big\|_p$$
$$\leqslant \Big\|\Big(\sum_{j\in\mathbb{Z}} \Big(\sum_{k=j}^{\infty} 2^{(j-k)(mr-n+\alpha r)}\, 2^{k\alpha r} M\big(|\psi_k * f|^r\big)\Big)^{q/r}\Big)^{1/q}\Big\|_p$$

and therefore, as $mr - n + \alpha r > mr - \lambda r + \alpha r > 0$, we have by Proposition B.3

$$\|f\|_{\dot{F}^{\alpha}_{p,q}} \leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}} \big(2^{k\alpha r} M(|\psi_k * f|^r)\big)^{q/r}\Big)^{1/q}\Big\|_p.$$

Finally, since we have chosen $0 < r < \min\{q,p\}$, the vector-valued maximal inequality, Theorem 1.4, then gives (56).

It only remains to prove the Besov-Lipschitz case, (52). Choose $0 < r < p \leqslant 1$ such that $\lambda > n/r$. A similar argument to that used to prove (56) shows that

$$\|f\|_{\dot{B}^{\alpha}_{p,q}} \leqslant C\Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}\|M(|\psi_j * f|^r)\|_{p/r}^{1/r}\big)^q\Big)^{1/q}.$$

Since $0 < r < p$, (52) now follows from the scalar-valued maximal inequality.

□

Note that, if we had some way of proving the maximal function $M_{\lambda,m}(x,j)$ was finite, then the above theorem would generalise Theorem 3.7 to all $0 < p, q \leqslant \infty$. If we make the assumption $\psi \in \mathcal{S}$ (or, on the other hand, to assume that $f \in \dot{B}^{\alpha}_{\infty,\infty}$) then the finiteness of $M_{\lambda,m}(x,j)$ is fairly straight forward to prove. This follows from the result that every $f \in \mathcal{S}'$ is bounded above by a finite number of Schwartz norms (while the condition $f \in \dot{B}^{\alpha}_{\infty,\infty}$ could be exploited by using Theorem 2.5). However, even without these assumptions, we can still prove the case $p = 1$ by using the Calderón reproducing formula of Lemma 3.2.

COROLLARY 3.11. *Let $0 < q \leqslant \infty$, $\alpha \in \mathbb{R}$, and take $\lambda, m, \ell \in \mathbb{N}$ such that $\lambda > \max\{n, n/q\}$, and $m > \lambda - \alpha$. Suppose $\psi \in L^1$ satisfies:*

(i) The Tauberian condition;

(ii) The differentiability condition $\psi \in C^m$ *and* $\widehat{\psi} \in C^{n+1+\lambda+m}\big(\mathbb{R}^n \setminus \{0\}\big)$*;*

(iii) The decay condition $(1 + |\cdot|)^\ell \psi(\cdot) \in L^1$ *and*

$$|D^\kappa \psi(x)| \leqslant \frac{C}{(1+|x|)^\lambda}$$

for every $x \in \mathbb{R}^n$ *and any* $|\kappa| = m$*.*

Assume $f \in \mathcal{S}'$ *and* $D^\kappa f$ *is a bounded distribution for each* $|\kappa| = \ell$*. If, for each* $j \in \mathbb{Z}$*, the distribution* $\psi_j * f$ *is a continuous, locally integrable function, then there exists a constant* C *independent of* f *such that*

$$\|f\|_{\dot{F}^{\alpha}_{1,q}} \leqslant C\Big\| \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha}|\psi_k * f|)^q \Big)^{1/q} \Big\|_1. \tag{56}$$

Proof. The main difficulties were contained in Theorem 3.9 and Theorem 3.10, it only remains to prove that $M_{\lambda,m}(x,j) < \infty$. We proceed by assuming that

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} (2^{j\alpha}|\psi_j * f|)^q \Big)^{1/q} \Big\|_1 < \infty,$$

since otherwise there is nothing to prove. Using the monotone nature of the l^q spaces we have

$$\sup_{j\in\mathbb{Z}} 2^{j\alpha}\|\psi_j * f\|_1 < \infty.$$

Therefore, using the Calderón formula of Lemma 3.2 together with Lemma 3.8, we see that, for every $x \in \mathbb{R}^n$,

$$\begin{aligned}
|\psi_j * f(x)| &\leqslant |\phi_j * \psi_j * f(x)| + \sum_{k=j+1}^{\infty} |\psi_j * \eta_k * \psi_k * f(x)| \\
&\leqslant \|\phi_j\|_\infty \|\psi_j * f\|_1 + \sum_{k=j+1}^{\infty} \|\psi_j * \eta_k\|_\infty \|\psi_k * f\|_1 \\
&\leqslant C2^{j(n-\alpha)} + C\sum_{k=j+1}^{\infty} 2^{(j-k)m}2^{jn}2^{-k\alpha} \\
&\leqslant C2^{j(n-\alpha)} + C2^{j(n-\alpha)}\sum_{k=j+1}^{\infty} 2^{(j-k)(m+\alpha)}.
\end{aligned}$$

Thus as $m + \alpha > \max\{n, n/q\} \geqslant 0$ there exists a constant C independent of j (but depending on f) such that

$$|\psi_j * f(x)| \leqslant C2^{j(n-\alpha)}.$$

Therefore, for every $x, y \in \mathbb{R}^n$,

$$\frac{|\psi_k * f(y)|}{(1+2^j|x-y|)^\lambda} 2^{(j-k)m} \leqslant C2^{(j-k)(m+\alpha-n)}2^{j(n-\alpha)}$$

and thus taking the supremum over $y \in \mathbb{R}^n$ and $k \geqslant j$ we obtain

$$M_{\lambda,m}(x,j) \leqslant C2^{j(n-\alpha)} < \infty.$$

The result then follows from Theorem 3.10.

$\square$

CHAPTER 4

Complete Characterisations

We have now reached a point where we can completely characterise the spaces $\dot{B}^{\alpha}_{p,q}$ and $\dot{F}^{\alpha}_{p,q}$ without the assumption $\psi \in \mathcal{S}$. Suppose $\alpha \in \mathbb{R}$ and $m, \lambda \in \mathbb{N}$. We assume the kernel ψ satisfies the following conditions:

(I) ψ has $[\alpha]$ vanishing moments and satisfies the Tauberian condition,

(II) $\widehat{\psi} \in C^{n+1+\lambda}\big(\mathbb{R}^n \setminus \{0\}\big)$ and $\psi \in C^m$,

(III) ψ satisfies the decay conditions $(1+|\cdot|)^{[\alpha]+\lambda+1}\psi(\cdot) \in L^1$ and, for every $|\kappa| \leqslant m$,

$$(1+|\cdot|)^{\lambda} D^{\kappa}\psi(\cdot) \in L^1.$$

Then we have the following characterisation.

Theorem 4.1. *Let $0 < p, q \leqslant \infty$ and $\alpha \in \mathbb{R}$. Assume ψ satisfies (I), (II), and (III) with $\lambda > n/p$ and $m > \lambda - \alpha$. Then for any $f \in \mathcal{S}'$ the following statements are equivalent:*

(i) The distribution $f \in \dot{B}^{\alpha}_{p,q}$.

(ii) There exists a polynomial ρ such that $D^{\kappa}(f-\rho)$ is a bounded distribution for every $|\kappa| = [\alpha]+1$ and

$$\bigg(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}\|\psi_j * (f-\rho)\|_{H^p}\big)^q\bigg)^{1/q} < \infty.$$

Proof. If (i) holds then Theorem 2.7 together with Corollary 2.3 gives (ii). On the other hand if (ii) holds then Theorem 3.6 gives (i).

□

We remark that the above proof also showed that the norms $\|f\|_{\dot{B}^{\alpha}_{p,q}}$ and

$$\bigg(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}\|\psi_j * (f-\rho)\|_{H^p}\big)^q\bigg)^{1/q}$$

are equivalent. However, if we begin by assuming that $f \in \dot{B}^{\alpha}_{p,q}$, we can in fact prove a slightly stronger inequality. We should emphasis that the following corollary of Theorem 3.10 requires that we already have $f \in \dot{B}^{\alpha}_{p,q}$, and is therefore not a characterisation.

COROLLARY 4.2. *Let p, q, α, m, λ, and ψ be as in Theorem 4.1. Moreover assume $\widehat{\psi} \in C^{n+1+m+\lambda}(\mathbb{R}^n \setminus \{0\})$ and*

$$|D^\kappa \psi(x)| \leqslant \frac{C}{(1+|x|)^{n+1+\lambda}}$$

for any $|\kappa| = m$ and $x \in \mathbb{R}^n$. Then if $f \in \dot{B}^\alpha_{p,q}$ there exists a polynomial ρ such that

$$\left(\sum_{j\in\mathbb{Z}} \left(2^{j\alpha} \|\psi_j * (f-\rho)\|_{H^p}\right)^q \right)^{1/q} \leqslant C_1 \|f\|_{\dot{B}^\alpha_{p,q}} \leqslant C_2 \left(\sum_{j\in\mathbb{Z}} \left(2^{j\alpha} \|\psi_j * (f-\rho)\|_p\right)^q \right)^{1/q}. \tag{57}$$

PROOF. By Theorem 2.5, the conditions on ψ together with the assumption $f \in \dot{B}^\alpha_{p,q}$ implies that there exist a polynomial ρ such that the distribution $\psi_j * (f-\rho)$ is a bounded continuous function with

$$\|\psi_t * (f-\rho)\|_\infty \leqslant C t^{\alpha - n/p}. \tag{58}$$

Thus the statement of the corollary makes sense. The left hand side inequality of (57) follows immediately from Theorem 2.7. To prove the remaining inequality, we note that, using the assumption $f \in \dot{B}^\alpha_{p,q}$, we have determined that $\psi_j * (f-\rho)$ is a continuous function and moreover, Corollary 2.3 shows that $D^\kappa f$ is a bounded distribution for every $|\kappa| = [\alpha] + 1$. Thus by Theorem 3.10 it suffices to prove that the maximal function $M_{\lambda,m}(x,j)$ is finite. To this end by using (58) and letting $g = f - \rho$ we have

$$\frac{|\psi_k * g(y)|}{(1+2^j|x-y|)^\lambda} 2^{(j-k)m} \leqslant C 2^{(j-k)(m-n+\alpha)}\, 2^{j(n-\alpha)}.$$

Therefore, taking the supremum over $y \in \mathbb{R}^n$ and $k \geqslant j$, we obtain

$$M_{\lambda,m}(x,j) \leqslant C 2^{j(n-\alpha)} < \infty$$

as required. $\square$

We also have the Triebel-Lizorkin counterpart characterising the space $\dot{F}^\alpha_{p,q}$. Recall that we have defined

$$(\psi_k * f)^{**}_\lambda(x) = \sup_{y\in\mathbb{R}^n} \frac{|G_k * \psi_k * f(y)|}{(1+2^k|x-y|)^\lambda}$$

where G is the Gaussian kernel.

THEOREM 4.3. *Let $0 < p, q \leqslant \infty$, $p < \infty$, and $\alpha \in \mathbb{R}$. Assume ψ satisfies (I), (II), and (III) with $\lambda > \max\{n/p, n/q\}$ and $m > \lambda - \alpha$. Then for any $f \in \mathcal{S}'$ the following statements are equivalent:*

(i) The distribution $f \in \dot{F}^\alpha_{p,q}$.

(ii) There exists a polynomial ρ such that $D^\kappa(f-\rho)$ is a bounded distribution for every $|\kappa| = [\alpha]+1$ and

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \Big(2^{k\alpha} \big(\psi_k * (f-\rho)\big)^{**}_{\lambda} \Big)^q \Big)^{1/q} \Big\|_p < \infty.$$

PROOF. An application of Theorem 2.1, with ψ replaced with $G * \psi$, together with Corollary 2.3 shows that (i) implies (ii). Conversely, if (ii) holds, then Theorem 3.5 gives (i).

□

Note that, as remarked after the proof of Lemma 3.4, the use of the Gaussian kernel in the above characterisation is not essential. In fact, by considering the proof of Lemma 3.1, the Tauberian condition implies there exists $2a < b$ such that $|\widehat{\psi}(t\xi)| \geqslant c > 0$ for any $|\xi| = 1$ and $a \leqslant t \leqslant b$. Manipulating a few inequalities we can show we only require $\phi \in \mathcal{S}$ with $|\widehat{\phi}(\xi)| \geqslant c > 0$ for any $a/8 \leqslant |\xi| \leqslant 8b$ for the above theorem to hold with G replaced by ϕ.

As in the Besov-Lipschitz case we also have the following norm inequalities. Again we remark that the following is not a characterisation as we must make the assumption that $f \in \dot{F}^\alpha_{p,q}$.

COROLLARY 4.4. *Let p, q, α, m, λ, and ψ be as in Theorem 4.3. Moreover assume $\widehat{\psi} \in C^{n+1+m+\lambda}(\mathbb{R}^n \setminus \{0\})$ and*

$$|D^\kappa \psi(x)| \leqslant \frac{C}{(1+|x|)^{n+1+\lambda}}$$

for any $|\kappa| = m$ and $x \in \mathbb{R}^n$. Then if $f \in \dot{F}^\alpha_{p,q}$ there exists a polynomial ρ such that

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \big(2^{k\alpha} \psi^*_k(f-\rho)\big)^q \Big)^{1/q} \Big\|_p \leqslant C_1 \|f\|_{\dot{F}^\alpha_{p,q}} \leqslant C_2 \Big\| \Big(\sum_{k\in\mathbb{Z}} \big(2^{k\alpha} |\psi_k * (f-\rho)|\big)^q \Big)^{1/q} \Big\|_p \tag{59}$$

PROOF. The left hand side inequality in (59) follows immediately from Theorem 2.1. For the right hand side inequalities, by Theorem 3.10, we need only show that the maximal function $M_{\lambda,m}(x,j)$ is finite. As in the Besov-Lipschitz case this follows from Theorem 2.5. □

Although the Theorems 4.1 and 4.3 provided a complete characterisation of the spaces $\dot{B}^\alpha_{p,q}$ and $\dot{F}^\alpha_{p,q}$ they still require the use of a maximal type function. On the other hand, as we have seen in Theorem 3.7 and Corollary 3.11, it is possible to prove the above theorem without the need for a maximal function. Among the hypotheses required however, is firstly that we restrict our attention to case $p \geqslant 1$, and secondly that we make the far from satisfactory assumption that the distribution $\psi_j * f$ is a function. Note that both of these restrictions could be removed if we make the additional assumption that $f \in \mathcal{S}'$ satisfies

$$\|\mu_t * f\|_\infty \leqslant Ct^{\alpha - n/p} \tag{60}$$

for any $\mu \in \mathcal{S}$ with α vanishing moments. Since this assumption, together with Theorem 2.5, would then show that the distribution $\psi_j * f$ is a function and moreover that $M_{\lambda,m}(x,j) < \infty$. This would then allow us to apply Theorem 3.10 and thus characterise the spaces $\dot{B}^\alpha_{p,q}$ and $\dot{F}^\alpha_{p,q}$ without the need for a maximal function. However, the assumption (60) is essentially equivalent to the requirement that $f \in \dot{B}^{\alpha-n/p}_{\infty,\infty}$ and thus would not produce a particularly satisfying characterisation.

Despite the above remarks, when considering the Poison kernel we have seen that it is possible to define the convolution $\psi_j * f$ pointwise. Using this fact gives the following characterisation.

THEOREM 4.5. *Suppose $1 \leqslant p \leqslant \infty$, $0 < q \leqslant \infty$ and $\alpha \in \mathbb{R}$. Take $k \in \mathbb{N}$ such that $2k > \alpha + \lambda$ and let $\widehat{\psi} = |\xi|^{2k} e^{-2\pi|\xi|}$. Assume $f \in \mathcal{S}'$ and $D^\kappa f$ is a bounded distribution for each $|\kappa| = [\alpha] + 1$. Then if $\lambda > n/p$ there exists a polynomial ρ such that*

$$\Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha} \|\psi_j * (f - \rho)\|_{H^p} \big)^q \Big)^{1/q} \leqslant C_1 \|f\|_{\dot{B}^\alpha_{p,q}} \leqslant C_2 \Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha} \|\psi_j * f\|_p \big)^q \Big)^{1/q}.$$

Similarly, if $p < \infty$ and $\lambda > \max\{n/p, n/q\}$, we have

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \big(2^{k\alpha} \psi^*_k (f - \rho) \big)^q \Big)^{1/q} \Big\|_p \leqslant C_1 \|f\|_{\dot{F}^\alpha_{p,q}} \leqslant C_2 \Big\| \Big(\sum_{k\in\mathbb{Z}} \big(2^{k\alpha} |\psi_k * f| \big)^q \Big)^{1/q} \Big\|_p.$$

PROOF. The conditions on ψ imply that we have the estimate

$$|\psi(x)| \leqslant \frac{C}{(1+|x|)^{n+1+[\alpha]+1+\lambda}}$$

and moreover that the convolution $\psi_j * f$ is a well defined smooth function. See the discussion at the beginning of Chapter 3, Section 3. Thus the left hand side inequalities follow from Theorem 2.1 while the right hand inequalities follow from Theorem 3.7 and Theorem 3.11.

$\square$

Further Results and Concluding Remarks

The previous results all have a continuous formulation. In fact it should be possible to prove the following. For a kernel ψ satisfying similar conditions to the theorems presented earlier, there exists polynomials ρ_1 and ρ_2 such that

$$\left(\int_0^\infty \big(t^{-\alpha}\|\psi_t*(f-\rho_1)\|_{H^p}\big)^q\frac{dt}{t}\right)^{1/q} \leqslant C_1\|f\|_{\dot{B}^\alpha_{p,q}} \leqslant C_2\left(\int_0^\infty \big(t^{-\alpha}\|\psi_t*(f-\rho_2)\|_p\big)^q\frac{dt}{t}\right)^{1/q}$$

and

$$\left\|\left(\int_0^\infty \big(t^{-\alpha}|\psi_t^*(f-\rho_1)|\big)^q\frac{dt}{t}\right)^{1/q}\right\|_p \leqslant C_1\|f\|_{\dot{F}^\alpha_{p,q}} \leqslant C_2\left\|\left(\int_0^\infty \big(t^{-\alpha}|\psi_t*(f-\rho_2)|\big)^q\frac{dt}{t}\right)^{1/q}\right\|_p.$$

To prove the necessary direction, i.e. the left hand inequalities, we only need to write the integral as

$$\int_0^\infty = \sum_{k\in\mathbb{Z}}\int_{2^{k-1}}^{2^k},$$

the rest of the argument follows the proof of Theorem 2.1 almost exactly. For the sufficient direction matters are a little more complicated. In fact we need to first prove continuous versions of Lemma 3.3,

$$\varphi * f(x) = \int_0^\infty \mu_t * \psi_t * \varphi * f(x)\frac{dt}{t}$$

and Theorem 3.9,

$$|\psi_k^*(x)|^r \leqslant CM(|\psi_k * f|^r)(x) + \int_0^k M(|\psi_t * f|^r)(x)\frac{dt}{t}.$$

The proof would then proceed similarly to the proofs in [**5**].

Another extension of the results presented in this thesis would be to the inhomogeneous Besov-Lipschitz and Triebel-Lizorkin spaces. We will not consider this extension here and instead leave the formulation and proof to the interested reader.

APPENDIX A

Construction of φ

In this brief section we construct an example of the kernel φ used to define the spaces $\dot{B}^{\alpha}_{p,q}$ and $\dot{F}^{\alpha}_{p,q}$. We begin by noting the following elementary result.

LEMMA A.1. *The function*

$$h(t) = \begin{cases} e^{\frac{-1}{t}} & t > 0 \\ 0 & t \leqslant 0 \end{cases}$$

is smooth and rapidly decreasing.

The next lemma has been used implicitly throughout this thesis. It allows us to construct cutoff functions in $\mathcal{S}$ which are supported in some arbitrary annulus about the origin. Note that by translating the η in Lemma A.2 we can construct a function in $\mathcal{S}$ supported in an arbitrary annulus.

LEMMA A.2. *Given any* $0 < r_1 < a_1 < a_2 < r_2 < \infty$ *there exists* $\eta \in \mathcal{S}$ *such that*

$$\eta(x) = \begin{cases} 1 & a_1 \leqslant |x| \leqslant a_2 \\ 0 & |x| \leqslant r_1 \quad or \ |x| \geqslant r_2. \end{cases}$$

PROOF. Define

$$\eta_1(t) = \frac{h(r_2 - t)}{h(r_2 - t) + h(t - a_2)}$$

where h is the function from the previous lemma. Then since $h(t) > 0$ for $t > 0$ we see that $h(r_2 - t) + h(t - a_2) > 0$ for all $t \in \mathbb{R}$ and hence η_1 is smooth on $\mathbb{R}$. Moreover η_1 satisfies

$$\eta_1(t) = \begin{cases} 0 & t \geqslant r_2 \\ 1 & t \leqslant a_2. \end{cases}$$

Similarly we define

$$\eta_2(t) = \frac{h(t - r_1)}{h(t - r_1) + h(a_1 - t)}.$$

Then η_2 is smooth and moreover

$$\eta_2(t) = \begin{cases} 0 & t \leqslant r_1 \\ 1 & t \geqslant a_1. \end{cases}$$

Finally we let $\eta(x) = \eta_1(|x|)\eta_2(|x|)$. Now since each η_i is the composition of two smooth functions it is also smooth (at least for $x > 0$, since η_i is also zero in a neighbourhood of the origin, we also have smoothness at $x = 0$) and has compact support. Therefore $\eta \in \mathcal{S}$ and it is easy to see the required properties hold. $\square$

Using the above lemma we obtain $\eta \in \mathcal{S}$ such that $0 \leqslant \eta(\xi) \leqslant 1$ for all $\xi \in \mathbb{R}^n$, and moreover,

$$\eta(\xi) = \begin{cases} 1 & 3/4 \leqslant |\xi| \leqslant 3/2 \\ 0 & |\xi| \leqslant 1/2 \text{ or } |\xi| \geqslant 2. \end{cases}$$

Now take

$$\rho(\xi) = \Big(\sum_{j\in\mathbb{Z}} \eta(2^{-j}\xi)\eta(2^{-j}\xi)\Big)^{1/2}.$$

We note that $\rho(\xi) \geqslant 1$ for any $1/2 \leqslant |\xi| \leqslant 2$ and $\rho(2^{-j}\xi) = \rho(\xi)$. Finally we define

$$\widehat{\varphi}(\xi) = \frac{\eta(\xi)}{\rho(\xi)}.$$

Then supp $\widehat{\varphi} \subseteq \{1/2 \leqslant |\xi| \leqslant 2\}$ and we have, for any $\xi \neq 0$,

$$\sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^{-j}\xi)\widehat{\varphi}(2^{-j}\xi) = \frac{1}{\rho(\xi)^2}\sum_{j\in\mathbb{Z}} \eta(2^{-j}\xi)\eta(2^{-j}\xi) = 1.$$

Therefore we have constructed a $\varphi \in \mathcal{S}$ with the required properties.

APPENDIX B

Technical Results

We now present a few technical results which were used throughout this thesis, the proofs to Theorem B.1 and Propositions B.2 and B.3 originated in [**6**]. The authors of [**6**] noted that these results seem to be well-known, however a proof or even a reference to a proof is rare. We will reprove these results here for the reader's convenience.

THEOREM B.1. *Suppose $\mu \in \mathcal{S}$ has k vanishing moments. Then for every $|\kappa| = k$ there exists $\mu^\kappa \in \mathcal{S}$ such that*

$$\mu = \sum_{|\kappa|=k+1} D^\kappa \mu^\kappa.$$

PROOF. We proceed via induction on the dimension n. Suppose firstly that $n = 1$ and μ has k vanishing moments. For $j = 1...k+1$ let

$$\mu^{(j)}(x) = \frac{1}{(j-1)!} \int_{-\infty}^{x} (x-s)^{j-1} \mu(s) ds.$$

Lebesgue's Differentiation Theorem then gives $\partial_x \mu^{(j)} = \mu^{(j-1)}$ with $\mu^{(0)} = \mu$, hence $\partial_x^{k+1} \mu^{(k+1)} = \mu$. It is straight forward to verify $\mu^{(j)} \in \mathcal{S}$, we only have to keep in mind that the vanishing moments of μ imply that

$$\int_{-\infty}^{x} (x-s)^{j-1} \mu(s) ds = -\int_{x}^{\infty} (x-s)^{j-1} \mu(s) ds.$$

Thus the result follows when $n = 1$.

Assume result holds for dimension $n-1$ and write $x = (x', x_n)$ for $x \in \mathbb{R}^n$, with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. For $j = 0, ..., k$ let

$$g^{(j)}(x') = \int_{\mathbb{R}} x_n^j \mu(x', x_n) dx_n.$$

Then $g^{(j)} \in \mathcal{S}(\mathbb{R}^{n-1})$ and moreover the vanishing moments of μ imply that $g^{(j)}$ has $k-j$ vanishing moments.

Take $\phi \in \mathcal{S}(\mathbb{R})$ such that $\widehat{\phi}(\xi) = 1$ for $|\xi| \leqslant 1$. Since $\int_{\mathbb{R}} \phi(t)\, dt = \widehat{\phi}(0)$ and $\int_{\mathbb{R}} (-2\pi i t)^j \phi(t) e^{-2\pi i t \xi}\, dt = \partial_\xi^j \widehat{\phi}(\xi)$ we have

$$\int_{\mathbb{R}} \phi(t) t^j dt = \begin{cases} 1 & j = 0 \\ 0 & j \geqslant 1 \end{cases}$$

and hence integration by parts gives

$$\int_{\mathbb{R}} \partial_t^m \phi(t) t^j dt = \begin{cases} (-1)^j j\,! & m = j \\ 0 & m < j. \end{cases}$$

Let

$$r(x', x_n) = \mu(x', x_n) - \sum_{j=0}^{k} \frac{(-1)^j}{j\,!} g^{(j)}(x') \partial_{x_n}^j \phi(x_n).$$

Then $r \in \mathcal{S}(\mathbb{R}^n)$, $r(x', \cdot) \in \mathcal{S}(\mathbb{R})$ and moreover $\int_{\mathbb{R}^n} x_n^j r(x', x_n) dx_n = 0$ for $j = 0, ..., k$. Therefore, as in the one dimensional case, letting

$$h(x', x_n) = \frac{1}{k!} \int_{-\infty}^{x_n} (x_n - s)^k r(x', s) ds$$

we have $h(x', \cdot) \in \mathcal{S}(\mathbb{R})$ and $r(x', x_n) = \partial_{x_n}^{k+1} h(x', x_n)$.

Now by definition each $g^{(j)} \in \mathcal{S}(\mathbb{R}^{n-1})$ has $k - j$ vanishing moments. Thus by the induction hypothesis there exist $h_{(j)}^{\alpha'} \in \mathcal{S}(\mathbb{R}^{n-1})$ such that

$$g^{(j)}(x') = \sum_{|\alpha'| = k-j+1} D^{\alpha'} h_{(j)}^{\alpha'}(x'),$$

where α' is a $n-1$ dimensional multi-index with $|\alpha'| = k - j + 1$. Therefore

$$\begin{aligned} \mu(x', x_n) &= r(x', x_n) + \sum_{j=0}^{k} \frac{(-1)^j}{j\,!} g^{(j)}(x') \partial_{x_n}^j \phi(x_n) \\ &= \partial_{x_n}^{k+1} h(x', x_n) + \sum_{j=0}^{k} \frac{(-1)^j}{j\,!} \sum_{|\alpha'| = k-j+1} D^{\alpha'} h_{(j)}^{\alpha'}(x') \partial_{x_n}^j \phi(x_n) \\ &= \sum_{|\alpha| = k+1} D^{\alpha} \mu^{\alpha}(x', x_n) \end{aligned}$$

with $\alpha = (\alpha', j)$ and

$$\mu^{(\alpha', j)}(x) = \begin{cases} h(x', x_n) & j = k+1 \\ \frac{(-1)^j}{j\,!} h_{(j)}^{\alpha'}(x') \phi(x_n) & j < k+1. \end{cases}$$

□

We remark that the above theorem can be generalised to include functions not in $\mathcal{S}$. In fact we can replace the assumption $\mu \in \mathcal{S}$ with $\mu \in C^0$ and

$$|\mu(x)| \leqslant \frac{C}{(1 + |x|)^{n+k+1}}.$$

However we can now only conclude that $\mu^{\kappa} \in C^{k+1}$. The proof of this generalisation is a straight forward extension of the above argument.

Proposition B.2. *Fix $0 < p, q \leqslant \infty$ and let $\delta = \min\{p, q, 1\}$. Then if $\{a_j\} \in l^\delta(\mathbb{Z})$ and f_k is a sequence of measurable functions we have*

$$\Big(\sum_{k\in\mathbb{Z}} \big(\big\|\sum_{j\in\mathbb{Z}} a_{j-k} f_j\big\|_p\big)^q\Big)^{1/q} \leqslant \Big(\sum_{j\in\mathbb{Z}} |a_j|^\delta\Big)^{1/\delta}\Big(\sum_{j\in\mathbb{Z}} \big\|f_j\big\|_p^q\Big)^{1/q}.$$

Proof. The proof is based on Young's Inequality and the inequality

$$\Big(\sum_{j\in\mathbb{Z}} |b_j|\Big)^r \leqslant \sum_{j\in\mathbb{Z}} |b_j|^r$$

which holds whenever $0 < r \leqslant 1$. We proceed by dividing the proof into cases. First suppose $p, q \geqslant 1$. Then since

$$\|\sum_{k\in\mathbb{Z}} a_{j-k} f_k\|_p \leqslant \sum_{k\in\mathbb{Z}} \|a_{j-k} f_k\|_p$$

we can use Young's Inequality for sequences to obtain

$$\begin{aligned}\Big(\sum_{k\in\mathbb{Z}} \big(\big\|\sum_{j\in\mathbb{Z}} a_{j-k} f_j\big\|_p\big)^q\Big)^{1/q} &\leqslant \Big(\sum_{k\in\mathbb{Z}} \big(\sum_{j\in\mathbb{Z}} \|a_{j-k} f_j\|_p\big)^q\Big)^{1/q} \\ &\leqslant \Big(\sum_{j\in\mathbb{Z}} |a_j|\Big)\Big(\sum_{j\in\mathbb{Z}} \big\|f_j\big\|_p^q\Big)^{1/q}.\end{aligned}$$

Similarly if $p \geqslant 1$ and $q \leqslant 1$ we have

$$\begin{aligned}\Big(\sum_{k\in\mathbb{Z}} \big(\big\|\sum_{j\in\mathbb{Z}} a_{j-k} f_j\big\|_p\big)^q\Big)^{1/q} &\leqslant \Big(\sum_{k\in\mathbb{Z}} \big(\sum_{j\in\mathbb{Z}} \|a_{j-k} f_j\|_p\big)^q\Big)^{1/q} \\ &\leqslant \Big(\sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \|a_{j-k} f_j\|_p^q\Big)^{1/q} \\ &\leqslant \Big(\sum_{j\in\mathbb{Z}} |a_j|^q\Big)^{1/q}\Big(\sum_{k\in\mathbb{Z}} \|f_j\|_p^q\Big)^{1/q}.\end{aligned}$$

Thus we have proven, for any $1 \leqslant p \leqslant \infty$ and $0 < q \leqslant \infty$,

$$\Big(\sum_{k\in\mathbb{Z}} \big(\big\|\sum_{j\in\mathbb{Z}} a_{j-k} f_j\big\|_p\big)^q\Big)^{1/q} \leqslant \Big(\sum_{j\in\mathbb{Z}} |a_j|^\delta\Big)^{1/\delta}\Big(\sum_{j\in\mathbb{Z}} \|f_j\|_p^q\Big)^{1/q}.$$

Now assume $0 < p < 1$. Then

$$\Big(\sum_{k\in\mathbb{Z}} \big(\big\|\sum_{j\in\mathbb{Z}} a_{j-k} f_j\big\|_p\big)^q\Big)^{1/q} \leqslant \Big(\sum_{k\in\mathbb{Z}} \big(\sum_{j\in\mathbb{Z}} \|a_{j-k} f_j\|_p^p\big)^{q/p}\Big)^{1/q}.$$

If $q \geqslant p$ then since $q/p \geqslant 1$ we can apply Young's Inequality for sequences to obtain

$$\Big(\sum_{k\in\mathbb{Z}} \big(\big\|\sum_{j\in\mathbb{Z}} a_{j-k} f_j\big\|_p\big)^q\Big)^{1/q} \leqslant \Big(\sum_{j\in\mathbb{Z}} |a_j|^p\Big)^{1/p}\Big(\sum_{j\in\mathbb{Z}} \big\|f_j\big\|_p^q\Big)^{1/q}.$$

On the other hand if $0 < q < p < 1$ we have

$$\Big(\sum_{k\in\mathbb{Z}}(\|\sum_{j\in\mathbb{Z}}a_{j-k}f_j\|_p)^q\Big)^{1/q} \leqslant \Big(\sum_{k\in\mathbb{Z}}(\sum_{j\in\mathbb{Z}}\|a_{j-k}f_j\|_p^p)^{q/p}\Big)^{1/q}$$
$$\leqslant \Big(\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\|a_{j-k}f_j\|_p^q\Big)^{1/q}$$
$$\leqslant \Big(\sum_{j\in\mathbb{Z}}|a_j|^q\Big)^{1/q}\Big(\sum_{j\in\mathbb{Z}}\|f_j\|_p^q\Big)^{1/q}.$$

□

PROPOSITION B.3. *Fix $0 < p < \infty$, $0 < q \leqslant \infty$ and let $\delta = \min\{1, q\}$. If $(a_k)_{k\in\mathbb{Z}} \in l^q$ and $(f_k)_{k\in\mathbb{Z}}$ is a sequence of measurable functions we have*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}(\sum_{k\in\mathbb{Z}}|a_{j-k}f_k|)^q\Big)^{1/q}\Big\|_p \leqslant \Big(\sum_{j\in\mathbb{Z}}|a_j|^\delta\Big)^{1/\delta}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_p$$

PROOF. The case $q \geqslant 1$ follows immediately from Young's Inequality for sequences. Suppose $0 < q < 1$. Then since

$$\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|a_{j-k}f_k(x)|\Big)^q \leqslant \Big(\sum_{j\in\mathbb{Z}}|a_j|^q\Big)\Big(\sum_{k\in\mathbb{Z}}|f_k(x)|^q\Big)$$

we have

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}(\sum_{k\in\mathbb{Z}}|a_{j-k}f_k|)^q\Big)^{1/q}\Big\|_p \leqslant \Big(\sum_{j\in\mathbb{Z}}|a_j|^\delta\Big)^{1/\delta}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_p$$

as required. □

LEMMA B.4. *Let $m \in \mathbb{N}$ and $r > 0$. Suppose $g \in C^m$ and for every $x \in \mathbb{R}^n$ and any $|\kappa| = m$ we have the estimate*

$$|D^\kappa g(x)| \leqslant C(1 + |x|)^r.$$

Then for any $x \in \mathbb{R}^n$ we have

$$|g(x)| \leqslant C(1 + |x|)^{r+m}.$$

PROOF. We proceed via induction. Suppose first that $m = 1$ and for $t \in \mathbb{R}$ let $h(t) = g(tx)$. Then by the mean value theorem we have

$$|h(1) - h(0)| \leqslant |\partial_t h(t)|$$

for some $0 \leqslant t \leqslant 1$. Now since

$$|\partial_t h(t)| \leqslant \sum_{j=1}^n |x_j\partial_{x_j}g(tx)|$$
$$\leqslant C|x|(1 + |x|)^r$$

we have

$$|g(x)| \leqslant |g(0)| + |x|(1+|x|)^r \leqslant C(1+|x|)^{r+1}$$

and hence the case $m = 1$ holds.

Now suppose result holds for m and we have $g \in C^{m+1}$ with

$$|D^\kappa g(x)| \leqslant C(1+|x|)^r$$

for every $|\kappa| = m+1$. Take any multi-index α such that $|\alpha| = m$. Then for any $i = 1, ..., n$ we have

$$|\partial_{x_i} D^\alpha g(x)| \leqslant C(1+|x|)^r$$

and therefore by the case $m = 1$ proved earlier we see that

$$|D^\alpha g(x)| \leqslant C(1+|x|)^{r+1}.$$

As α was arbitrary this holds for every $|\alpha| = m$. Thus by the induction hypothesis we obtain

$$|g(x)| \leqslant C(1+|x|)^{r+1+m}$$

and therefore result follows. $\square$

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